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ALGEBRA

PART I.

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PREFACE.

THIS Book is published at the request of many Teachers. It is intended to assist, *first,* Masters who cannot devote much time to the examination of the work of their pupils, and *secondly,* Students who are unable to obtain the help of competent instructors. In working the Exercises, the simplest and most obvious methods of solution have in all cases been adopted.

J. HAMBLIN SMITH.

CAMBRIDGE, *May* 1874.

ELEMENTARY ALGEBRA.

KEY.

IV.

1. $a+b+3a-2b=4a-b.$

2. $a+b-a+3b=4b.$

3. $3a+5b-6c-2a-4b+2c=a+b-4a.$

4. $a+b-c-a+b+c=2b.$

5. $14x-5x+9-\{4-3x-2x+3\}=14x-5x+9-4+3x+2x-3=14x+2.$

6. $4x-\{3x-(2x-x+a)\}=4x-\{3x-2x+x-a\}=4x-3x+2x-x+a$
$=2x+a.$

7. $15x-\{7x+(3x+a-x)\}=15x-\{7x+3x+a-x\}$
$=15x-7x-3x-a+x=6x-a.$

8. $a-[b+\{a-b-a\}]=a-[b+a-b-a]=a-b-a+b+a=a.$

9. $6a+[4a-\{8b-2a-4b-22b\}-7b]-[7b+\{8a-3b-4a+8b\}+6a]$
$=6a+[4a-8b+2a+4b+22b-7b]-[7b+8a-3b-4a+8b+6a]$
$=6a+4a-8b+2a+4b+22b-7b-7b-8a+3b+4a-8b-6a$
$=2a-b.$

10. $b-[b-a-b-\{b-(b-a+b)\}]=b-[b-a-b-\{b-b+a-b\}]$
$=b-[b-a-b-b+b-a+b]=b-b+a+b+b-b+a-b=2a.$

11. $2c-6a+b-\{c-5a-2b-a+3b\}=2c-6a+b-c+5a+2b+a-3b=c.$

12. $2x-\{a-(2a-[3a-(4a-[5a-6a+x])])\}$
$=2x-\{a-(2a-[3a-(4a-5a+6a-x)])\}$
$=2x-\{a-(2a-[3a-4a+5a-6a+x])\}$
$=2x-\{a-(2a-3a+4a-5a+6a-x)]\}$
$=2x-\{a-2a+3a-4a+5a-6a+x\}$
$=2x-a+2a-3a+4a-5a+6a-x=x+3a.$

13. $25a-19b-[3b-\{4a-5b+6c\}]=25a-19b-[3b-4a+5b-6c]$
$=25a-19b-3b+4a-5b+6c=29a-27b+6c.$

VIII.

1.
$$\begin{array}{lll} x & +3 & \\ x & +9 & \\ \hline x^2 & +3x & \\ & +9x & +27 \\ \hline x^2 & +12x & +27 \end{array}$$

2.
$$\begin{array}{lll} x & +15 & \\ x & -7 & \\ \hline x^2 & +15x & \\ & -7x & -105 \\ \hline x^2 & +8x & -105 \end{array}$$

3.
$$\begin{array}{lll} x & -12 & \\ x & +10 & \\ \hline x^2 & -12x & \\ & +10x & -120 \\ \hline x^2 & -2x & -120 \end{array}$$

4.
$$\begin{array}{lll} x & -8 & \\ x & -7 & \\ \hline x^2 & -8x & \\ & -7x & +56 \\ \hline x^2 & -15x & +56 \end{array}$$

5.
$$\begin{array}{lll} a & -3 & \\ a & -5 & \\ \hline a^2 & -3a & \\ & -5a & +15 \\ \hline a^2 & -8a & +15 \end{array}$$

6.
$$\begin{array}{lll} y & -6 & \\ y & +13 & \\ \hline y^2 & -6y & \\ & +13y & -78 \\ \hline y^2 & +7y & -78 \end{array}$$

7.
$$\begin{array}{lll} x^2 & -4 & \\ x^2 & +5 & \\ \hline x^4 & -4x^2 & \\ & +5x^2 & -20 \\ \hline x^4 & +x^2 & -20 \end{array}$$

8.
$$\begin{array}{lllll} x^2 & -6x & +9 & & \\ x^2 & -6x & +5 & & \\ \hline x^4 & -6x^3 & +9x^2 & & \\ & -6x^3 & +36x^2 & -54x & \\ & & +5x^2 & -30x & +45 \\ \hline x^4 & -12x^3 & +50x^2 & -84x & +45 \end{array}$$

9.
$$\begin{array}{l}
x^2+5x-3 \\
x^2-5x-3 \\
\hline
x^4+5x^3-3x^2 \\
\qquad -5x^3-25x^2+15x \\
\qquad\qquad -3x^2-15x+9 \\
\hline
x^4 \qquad -31x^2 \qquad +9
\end{array}$$

10.
$$\begin{array}{l}
a^3-3a+2 \\
a^3-3a^2+2 \\
\hline
a^6-3a^4+2a^3 \\
\qquad -3a^5+9a^3-6a^2 \\
\qquad\qquad +2a^3-6a+4 \\
\hline
a^6-3a^5-3a^4+13a^3-6a^2-6a+4
\end{array}$$

11.
$$\begin{array}{l}
x^2-x+1 \\
x^2+x-1 \\
\hline
x^4-x^3+x^2 \\
\qquad +x^3-x^2+x \\
\qquad\qquad -x^2+x-1 \\
\hline
x^4 \qquad -x^2+2x-1
\end{array}$$

12.
$$\begin{array}{l}
x^2+xy+y^2 \\
x^2-xy+y^2 \\
\hline
x^4+x^3y+x^2y^2 \\
\qquad -x^3y-x^2y^2-xy^3 \\
\qquad\qquad +x^2y^2+xy^3+y^4 \\
\hline
x^4 \qquad +x^2y^2 \qquad +y^4
\end{array}$$

13.
$$\begin{array}{l}
x^2+xy+y^2 \\
x-y \\
\hline
x^3+x^2y+xy^2 \\
\qquad -x^2y-xy^2-y^3 \\
\hline
x^3 \qquad\qquad -y^3
\end{array}$$

14.
$$\begin{array}{l}
a^2-x^2 \\
a^4+a^2x^2+x^4 \\
\hline
a^6-a^4x^2 \\
\qquad +a^4x^2-a^2x^4 \\
\qquad\qquad +a^2x^4-x^6 \\
\hline
a^6 \qquad\qquad -x^6
\end{array}$$

15.
$$\begin{array}{l}
x^3-3x^2+3x-1 \\
x^2+3x+1 \\
\hline
x^5-3x^4+3x^3-x^2 \\
\qquad +3x^4-9x^3+9x^2-3x \\
\qquad\qquad +x^3-3x^2+3x-1 \\
\hline
x^5 \qquad -5x^3+5x^2 \qquad -1
\end{array}$$

16.
$$\begin{array}{l}
x^3+3x^2y+9xy^2+27y^3 \\
x-3y \\
\hline
x^4+3x^3y+9x^2y^2+27xy^3 \\
\qquad -3x^3y-9x^2y^2-27xy^3-81y^4 \\
\hline
x^4 \qquad\qquad\qquad -81y^4
\end{array}$$

17.
$$\begin{array}{l}
a^3+2a^2b+4ab^2+8b^3 \\
a-2b \\
\hline
a^4+2a^3b+4a^2b^2+8ab^3 \\
\qquad -2a^3b-4a^2b^2-8ab^3-16b^4 \\
\hline
a^4 \qquad\qquad\qquad -16b^4
\end{array}$$

18.
$$\begin{array}{l}
8a^3+4a^2b+2ab^2+b^3 \\
2a-b \\
\hline
16a^4+8a^3b+4a^2b^2+2ab^3 \\
\qquad -8a^3b-4a^2b^2-2ab^3-b^4 \\
\hline
16a^4 \qquad\qquad\qquad -b^4
\end{array}$$

19.
$$\begin{array}{l}
a^3-2a^2b+3ab^2+4b^3\\
a^2-2ab-3b^2\\
\hline
a^5-2a^4b+3a^3b^2+4a^2b^3\\
\qquad -2a^4b+4a^3b^2-6a^2b^3-8ab^4\\
\qquad\qquad\quad -3a^3b^2+6a^2b^3-9ab^4-12b^5\\
\hline
a^5-4a^4b+4a^3b^2+4a^2b^3-17ab^4-12b^5
\end{array}$$

20.
$$\begin{array}{l}
a^3+3a^2b-2ab^2+3b^3\\
a^2+2ab-3b^2\\
\hline
a^5+3a^4b-2a^3b^2+3a^2b^3\\
\qquad +2a^4b+6a^3b^2-4a^2b^3+6ab^4\\
\qquad\qquad\quad -3a^3b^2-9a^2b^3+6ab^4-9b^5\\
\hline
a^5+5a^4b+a^3b^2-10a^2b^3+12ab^4-9b^5
\end{array}$$

21.
$$\begin{array}{l}
a^2-2ax+4x^2\\
a^2+2ax+4x^2\\
\hline
a^4-2a^3x+4a^2x^2\\
\qquad +2a^3x-4a^2x^2+8ax^3\\
\qquad\qquad\quad +4a^2x^2-8ax^3+16x^4\\
\hline
a^4 \qquad\quad +4a^2x^2 \qquad\quad +16x^4
\end{array}$$

22.
$$\begin{array}{l}
9a^2+3ax+x^2\\
9a^2-3ax+x^2\\
\hline
81a^4+27a^3x+9a^2x^2\\
\qquad -27a^3x-9a^2x^2-3ax^3\\
\qquad\qquad\quad +9a^2x^2+3ax^3+x^4\\
\hline
81a^4 \qquad\quad +9a^2x^2 \qquad\quad +x^4
\end{array}$$

23.
$$\begin{array}{l}
x^4-2ax^2+4a^2\\
x^4+2ax^2+4a^2\\
\hline
x^8-2ax^6+4a^2x^4\\
\qquad +2ax^6-4a^2x^4+8a^3x^2\\
\qquad\qquad\quad +4a^2x^4-8a^3x^2+16a^4\\
\hline
x^8 \qquad\quad +4a^2x^4 \qquad\quad +16a^4
\end{array}$$

24.
$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc\\
a\;+b\;+c\\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc\\
\qquad +a^2b+b^3\;+bc^2\;-ab^2-abc-b^2c\\
\qquad\qquad\quad +a^2c+b^2c+c^3-abc-ac^2-bc^2\\
\hline
a^3+b^3\;+c^3\;-3abc
\end{array}$$

25.
$$\begin{array}{llllll}
x^2+4xy & +5y^2 & & & & \\
x^3-3x^2y & -2xy^2 & +3y^3 & & & \\
\hline
x^5+4x^4y & +5x^3y^2 & & & & \\
\quad -3x^4y & -12x^3y^2 & -15x^2y^3 & & & \\
 & -2x^3y^2 & -8x^2y^3 & -10xy^4 & & \\
 & & +3x^2y^3 & +12xy^4 & +15y^5 & \\
\hline
x^5+x^4y & -9x^3y^2 & -20x^2y^3 & +2xy^4 & +15y^5 &
\end{array}$$

26.
$$\begin{array}{lllllll}
ab+cd+ac+bd & & & & & & \\
ab+cd-ac-bd & & & & & & \\
\hline
a^2b^2 & +abcd & +a^2bc & +ab^2d & & & \\
 & +abcd & +c^2d^2 & +ac^2d & +bcd^2 & & \\
 & & -a^2bc & -ac^2d & -a^2c^2 & -abcd & \\
 & & & -ab^2d & -bcd^2 & -abcd & -b^2d^2 \\
\hline
a^2b^2 & & +c^2d^2 & & -a^2c^2 & & -b^2d^2
\end{array}$$

27. $(x-a)(x+a)(x^2+a^2)(x^4+a^4)=(x^2-a^2)(x^2+a^2)(x^4+a^4)$
$=(x^4-a^4)(x^4+a^4)=x^8-a^8.$

28. $(x-a)(x+b)(x-c)=(x^2-ax+bx-ab)(x-c)$
$=x^3-ax^2+bx^2-cx^2-abx+acx-bcx+abc.$

29. $(1-x)(1+x)(1+x^2)(1+x^4)=(1-x^2)(1+x^2)(1+x^4)$
$=(1-x^4)(1+x^4)=1-x^8.$

30. First multiply x^2+xy+y^2 by $x-y$; the result is x^3-y^3.
Then multiply x^2-xy+y^2 by $x+y$; the result is x^3+y^3.
Then multiply x^3-y^3 by x^3+y^3; the result is x^6-y^6.

31. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^2-x^2)(a^2+x^2)(a^4+x^4)(a^8+x^8)$
$=(a^4-x^4)(a^4+x^4)(a^8+x^8)$
$=(a^8-x^8)(a^8+x^8)=a^{16}-x^{16}.$

32. $(x-5)(x-6)(x+7)=(x^2-11x+30)(x+7)=x^3-4x^2-47x+210$; therefore the coefficient of x is -47.

33. $(x+8)(x+3)(x-2)=(x^2+11x+24)(x-2)=x^3+9x^2+2x-48$;
therefore the coefficient of x is 2.

34. $(x-2)(x-3)(x+4)=(x^2-5x+6)(x+4)=x^3-x^2-14x+24$;
therefore the coefficient of x is -14.

35. $(x-a)(x-b)(x-c)=(x^2-ax-bx+ab)(x-c)$
$= x^3-ax^2-bx^2-cx^2+abx+acx+bcx-abc$;
therefore the coefficient of x is $ab+ac+bc$.

36. $(x^2+3x-2)(x^2-3x+2)(x^4-5)=(x^4-9x^2+12x-4)(x^4-5)$
$=x^8-9x^6+12x^5-9x^4+45x^2-60x+20$;
therefore the coefficient of x is -60.

37. $(x^2-x+1)(x^2+x-1)(x^4-x^2+1) = (x^4-x^2+2x-1)(x^4-x^2+1)$
$=x^8-2x^6+2x^5+x^4-2x^3+2x-1$;
therefore the coefficient of x is 2.

38. $(x^2-mx+1)(x^2-mx-1)(x^4-m^2x-1)$
$= (x^4-2mx^3+m^2x^2-1)(x^4-m^2x-1)=x^8-2mx^7+m^2x^6-m^2x^5$
$+2m^3x^4-2x^4-m^4x^3+2mx^3-m^2x^2+m^2x+1$;
therefore the coefficient of x is m^2.

IX.

8.
$$\begin{array}{l} -a^3-a^2-a \\ -a-1 \\ \hline a^4+a^3+a^2 \\ \qquad +a^3+a^2+a \\ \hline a^4+2a^3+2a^2+a \end{array}$$

9.
$$\begin{array}{l} 3x^2y-5xy^2+4y^3 \\ -2x-3y \\ \hline -6x^3y+10x^2y^2-8xy^3 \\ \qquad\qquad -9x^2y^2+15xy^3-12y^4 \\ \hline -6x^3y+x^2y^2+7xy^3-12y^4 \end{array}$$

10.
$$\begin{array}{l} -5m^2-6mn+7n^2 \\ -m+n \\ \hline 5m^3+6m^2n-7mn^2 \\ \qquad -5m^2n-6mn^2+7n^3 \\ \hline 5m^3+m^2n-13mn^2+7n^3 \end{array}$$

11.
$$\begin{array}{l} 13r^2-17r-45 \\ -r-3 \\ \hline -13r^3+17r^2+45r \\ \qquad -39r^2+51r+135 \\ \hline -13r^3-22r^2+96r+135 \end{array}$$

12. $$\begin{array}{l} 7x^3-8x^2z-9z^2 \\ -x-z \\ \hline -7x^4+8x^3z+9xz^2 \\ \qquad -7x^3z+8x^2z^2+9z^3 \\ \hline -7x^4+x^3z+8x^2z^2+9xz^2+9z^3 \end{array}$$

13. $$\begin{array}{l} -x^5+x^4y-x^3y^2 \\ -x-y \\ \hline x^6-x^5y+x^4y^2 \\ \quad +x^5y-x^4y^2+x^3y^3 \\ \hline x^6 \qquad\qquad\quad +x^3y^3 \end{array}$$

14. $$\begin{array}{l} -x^3-x^2y-xy^2-y^3 \\ -x-y \\ \hline x^4+x^3y+x^2y^2+xy^3 \\ \quad +x^3y+x^2y^2+xy^3+y^4 \\ \hline x^4+2x^3y+2x^2y^2+2xy^3+y^4 \end{array}$$

X.

12. $(x^2+2x-3)^2=x^4+4x^2+9+4x^3-6x^2-12x=x^4+4x^3-2x^2-12x+9.$

13. $(x^2-6x+7)^2=x^4+36x^2+49-12x^3+14x^2-84x$
$=x^4-12x^3+50x^2-84x+49.$

14. $(2x^2-7x+9)^2=4x^4+49x^2+81-28x^3+36x^2-126x$
$=4x^4-28x^3+85x^2-126x+81.$

16. $(x^4-4x^2y^2+y^4)^2=x^8+16x^4y^4+y^8-8x^6y^2+2x^4y^4-8x^2y^6$
$=x^8-8x^6y^2+18x^4y^4-8x^2y^6+y^8.$

27. $(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$
$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$

28. $(a-b-c)^3=(a-b)^3-3(a-b)^2c+3(a-b)c^2-c^3$
$=a^3-3a^2b+3ab^2-b^3-3a^2c+6abc-3b^2c+3ac^2-3bc^2-c^3.$

29. $(m+n)(m-n)(m+n)(m-n)=(m^2-n^2)(m^2-n^2)$
$=m^4-2m^2n^2+n^4.$

30. $(m^2+2mn+n^2)(m^2-n^2)=m^4+2m^3n+m^2n^2-m^2n^2-2mn^3-n^4$
$=m^4+2m^3n-2mn^3-n^4.$

XIV.

1.
$$\begin{array}{l} x+10)x^2+15x+50(x+5 \\ \underline{x^2+10x} \\ \qquad 5x+50 \\ \qquad \underline{5x+50} \end{array}$$

2.
$$\begin{array}{l} x-7)x^2-17x+70(x-10 \\ \underline{x^2-7x} \\ \qquad -10x+70 \\ \qquad \underline{-10x+70} \end{array}$$

3.
$$\begin{array}{l} x-3)x^2+x-12(x+4 \\ \underline{x^2-3x} \\ \qquad 4x-12 \\ \qquad \underline{4x-12} \end{array}$$

4.
$$\begin{array}{l} x+1)x^2+13x+12(x+12 \\ \underline{x^2+x} \\ \qquad 12x+12 \\ \qquad \underline{12x+12} \end{array}$$

5.
$$\begin{array}{l} x+6)x^3+13x^2+54x+72(x^2+7x+12 \\ \underline{x^3+6x^2} \\ \qquad 7x^2+54x \\ \qquad \underline{7x^2+42x} \\ \qquad\qquad 12x+72 \\ \qquad\qquad \underline{12x+72} \end{array}$$

6.
$$\begin{array}{l} x+1)x^3+x^2-x-1(x^2-1 \\ \underline{x^3+x^2} \\ \qquad -x-1 \\ \qquad \underline{-x-1} \end{array}$$

7.
$$\begin{array}{l} x+1)x^3+2x^2+2x+1(x^2+x+1 \\ \underline{x^3+x^2} \\ \qquad x^2+2x \\ \qquad \underline{x^2+x} \\ \qquad\qquad x+1 \\ \qquad\qquad \underline{x+1} \end{array}$$

8.
$$\begin{array}{l} x^2+3x+1)x^5-5x^3+7x^2+6x+1(x^3-3x^2+3x+1 \\ \underline{x^5+3x^4+x^3} \\ \quad -3x^4-6x^3+7x^2 \\ \quad \underline{-3x^4-9x^3-3x^2} \\ \qquad\qquad 3x^3+10x^2+6x \\ \qquad\qquad \underline{3x^3+9x^2+3x} \\ \qquad\qquad\qquad x^2+3x+1 \\ \qquad\qquad\qquad \underline{x^2+3x+1} \end{array}$$

9. $$\begin{array}{l} x^2-2x-1)x^4-4x^3+2x^2+4x+1(x^2-2x-1 \\ \qquad\qquad\underline{x^4-2x^3-x^2} \\ \qquad\qquad\quad -2x^3+3x^2+4x \\ \qquad\qquad\quad \underline{-2x^3+4x^2+2x} \\ \qquad\qquad\qquad\qquad -x^2+2x+1 \\ \qquad\qquad\qquad\qquad \underline{-x^2+2x+1} \end{array}$$

10. $$\begin{array}{l} x^2-2x+1)x^4-4x^3+6x^2-4x+1(x^2-2x+1 \\ \qquad\qquad\underline{x^4-2x^3+x^2} \\ \qquad\qquad\quad -2x^3+5x^2-4x \\ \qquad\qquad\quad \underline{-2x^3+4x^2-2x} \\ \qquad\qquad\qquad\qquad x^2-2x-1 \\ \qquad\qquad\qquad\qquad \underline{x^2-2x+1} \end{array}$$

11. $$\begin{array}{l} x^2+x-1)x^4-x^2+2x-1(x^2-x+1 \\ \qquad\qquad\underline{x^4+x^3-x^2} \\ \qquad\qquad\quad -x^3+2x-1 \\ \qquad\qquad\quad \underline{-x^3-x^2+x} \\ \qquad\qquad\qquad\qquad x^2+x-1 \\ \qquad\qquad\qquad\qquad \underline{x^2+x-1} \end{array}$$

12. $$\begin{array}{l} x+2)x^4-4x^2+8x+16(x^3-2x^2+8 \\ \qquad\quad\underline{x^4+2x^3} \\ \qquad\quad\; -2x^3-4x^2 \\ \qquad\quad\; \underline{-2x^3-4x^2} \\ \qquad\qquad\qquad 8x+16 \\ \qquad\qquad\qquad \underline{8x+16} \end{array}$$

$$\begin{array}{l}
13.\ x+4y)x^3+4x^2y+3xy^2+12y^3(x^2+3y^2 \\
\qquad\qquad \underline{x^3+4x^2y} \\
\qquad\qquad\qquad\qquad 3xy^2+12y^3 \\
\qquad\qquad\qquad\qquad \underline{3xy^2+12y^3}
\end{array}$$

$$\begin{array}{l}
14.\ a+b)a^4+4a^3b+6a^2b^2+4ab^3+b^4(a^3+3a^2b+3ab^2+b^3 \\
\qquad\quad \underline{a^4+a^3b} \\
\qquad\qquad 3a^3b+6a^2b^2 \\
\qquad\qquad \underline{3a^3b+3a^2b^2} \\
\qquad\qquad\qquad 3a^2b^2+4ab^3 \\
\qquad\qquad\qquad \underline{3a^2b^2+3ab^3} \\
\qquad\qquad\qquad\qquad ab^3+b^4 \\
\qquad\qquad\qquad\qquad \underline{ab^3+b^4}
\end{array}$$

$$\begin{array}{l}
15.\ a-b)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5(a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
\qquad\quad \underline{a^5-a^4b} \\
\qquad\qquad -4a^4b+10a^3b^2 \\
\qquad\qquad \underline{-4a^4b+4a^3b^2} \\
\qquad\qquad\qquad 6a^3b^2-10a^2b^3 \\
\qquad\qquad\qquad \underline{6a^3b^2-6a^2b^3} \\
\qquad\qquad\qquad\quad -4a^2b^3+5ab^4 \\
\qquad\qquad\qquad\quad \underline{-4a^2b^3+4ab^4} \\
\qquad\qquad\qquad\qquad\qquad ab^4-b^5 \\
\qquad\qquad\qquad\qquad\qquad \underline{ab^4-b^5}
\end{array}$$

$$\begin{array}{l}
16.\ x^2-6x+9)x^4-12x^3+50x^2-84x+45(x^2-6x+5 \\
\qquad\qquad\quad \underline{x^4-6x^3+9x^2} \\
\qquad\qquad\qquad -6x^3+41x^2-84x \\
\qquad\qquad\qquad \underline{-6x^3+36x^2-54x} \\
\qquad\qquad\qquad\qquad\quad 5x^2-30x+45 \\
\qquad\qquad\qquad\qquad\quad \underline{5x^2-30x+45}
\end{array}$$

17. $a^2 - 2ab - 3b^2)\, a^5 - 4a^4b + 4a^3b^2 + 4a^2b^3 - 17ab^4 - 12b^5\, (a^3 - 2a^2b + 3ab^2 + 4b^3$

$\underline{a^5 - 2a^4b - 3a^3b^2}$

$-2a^4b + 7a^3b^2 + 4a^2b^3$

$\underline{-2a^4b + 4a^3b^2 + 6a^2b^3}$

$3a^3b^2 - 2a^2b^3 - 17ab^4$

$\underline{3a^3b^2 - 6a^2b^3 - 9ab^4}$

$4a^2b^3 - 8ab^4 - 12b^5$

$\underline{4a^2b^3 - 8ab^4 - 12b^5}$

18. $2ax^2 - 3a^2x + a^3)\, 4a^2x^4 - 12a^3x^3 + 13a^4x^2 - 6a^5x + a^6\, (2ax^2 - 3a^2x + a^3$

$\underline{4a^2x^4 - 6a^3x^3 + 2a^4x^2}$

$-6a^3x^3 + 11a^4x^2 - 6a^5x$

$\underline{-6a^3x^3 + 9a^4x^2 - 3a^5x}$

$2a^4x^2 - 3a^5x + a^6$

$\underline{2a^4x^2 - 3a^5x + a^6}$

19. $x^2 + x - 1)\, x^4 - x^2 + 2x - 1\, (x^2 - x + 1$

$\underline{x^4 + x^3 - x^2}$

$-x^3 + 2x - 1$

$\underline{-x^3 - x^2 + x}$

$x^2 + x - 1$

$\underline{x^2 + x - 1}$

20. $x^2 + 2a^2)\, x^4 + a^2x^2 - 2a^4\, (x^2 - a^2$

$\underline{x^4 + 2a^2x^2}$

$-a^2x^2 - 2a^4$

$\underline{-a^2x^2 - 2a^4}$

21. $x - 15y)\, x^2 - 13xy - 30y^2\, (x + 2y$

$\underline{x^2 - 15xy}$

$2xy - 30y^2$

$\underline{2xy - 30y^2}$

22. $x + y)\, x^5 + y^5\, (x^4 - x^3y + x^2y^2 - xy^3 + y^4$

$\underline{x^5 + x^4y}$

$-x^4y + y^5$

$\underline{-x^4y - x^3y^2}$

$x^3y^2 + y^5$

$\underline{x^3y^2 + x^2y^3}$

$-x^2y^3 + y^5$

$\underline{-x^2y^3 - xy^4}$

$xy^4 + y^5$

$\underline{xy^4 + y^5}$

23. $x-y)x^6-y^6(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$

$$\begin{array}{l}
\underline{x^6-x^5y} \\
x^5y-y^6 \\
\underline{x^5y-x^4y^2} \\
x^4y^2-y^6 \\
\underline{x^4y^2-x^3y^3} \\
x^3y^3-y^6 \\
\underline{x^3y^3-x^2y^4} \\
x^2y^4-y^6 \\
\underline{x^2y^4-xy^5} \\
xy^5-y^6 \\
\underline{xy^5-y^6}
\end{array}$$

24. $a-b+c)a^2-b^2+2bc-c^2(a+b-c$

$$\begin{array}{l}
\underline{a^2-ab+ac} \\
ab-b^2-ac+2bc-c^2 \\
\underline{ab-b^2+bc} \\
-ac+bc-c^2 \\
\underline{-ac+bc-c^2}
\end{array}$$

25. $-1+b)b-3b^2+3b^3-b^4(-b+2b^2-b^3$

$$\begin{array}{l}
\underline{b-b^2} \\
-2b^2+3b^3 \\
\underline{-2b^2+2b^3} \\
b^3-b^4 \\
\underline{b^3-b^4}
\end{array}$$

26. $a+b-c-d)a^2-2ad+2bc-b^2-c^2+d^2(a-b+c-d$

$$\begin{array}{l}
\underline{a^2+ab-ac-ad} \\
-ab-b^2+ac-ad+2bc-c^2+d^2 \\
\underline{-ab-b^2+bc+bd} \\
ac+bc-bd-ad-c^2+d^2 \\
\underline{ac+bc-c^2-cd} \\
-ad-bd+cd+d^2 \\
\underline{ad-bd+cd+d^2}
\end{array}$$

27. $$\begin{array}{l}
x+y+z)\,x^3+y^3+z^3-3xyz\,(x^2-xy-xz+y^2-yz+z^2 \\
\underline{x^3+x^2y+x^2z} \\
-x^2y-x^2z-3xyz+y^3+z^3 \\
\underline{-x^2y-xy^2-xyz} \\
-x^2z+xy^2-2xyz+y^3+z^3 \\
\underline{-x^2z-xyz-xz^2} \\
xy^2-xyz+xz^2+y^3+z^3 \\
\underline{xy^2+y^3+y^2z} \\
-xyz-y^2z+xz^2+z^3 \\
\underline{-xyz-y^2z-yz^2} \\
xz^2+yz^2+z^3 \\
\underline{xz^2+yz^2+z^3}
\end{array}$$

28. $$\begin{array}{l}
x^3+y^2)\,x^{15}+y^{10}\,(x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8 \\
\underline{x^{15}+x^{12}y^2} \\
-x^{12}y^2+y^{10} \\
\underline{-x^{12}y^2-x^9y^4} \\
x^9y^4+y^{10} \\
\underline{x^9y^4+x^6y^6} \\
-x^6y^6+y^{10} \\
\underline{-x^6y^6-x^3y^8} \\
x^3y^8+y^{10} \\
\underline{x^3y^8+y^{10}}
\end{array}$$

29. $$\begin{array}{l}
p-q+3r)\,p^2+pq+2pr-2q^2+7qr-3r^2\,(p+2q-r \\
\underline{p^2-pq+3pr} \\
2pq-pr-2q^2+7qr \\
\underline{2pq-2q^2+6qr} \\
-pr+qr-3r^2 \\
\underline{-pr+qr-3r^2}
\end{array}$$

30.
$$\begin{array}{l}
a^4 + a^3b + a^2b^2 + ab^3 + b^4)\,a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8\,(a^4 - a^3b + a^2b^2 - ab^3 + b^4 \\
\underline{a^8 + a^7b + a^6b^2 + a^5b^3 + a^4b^4} \\
-a^7b - a^5b^3 + a^2b^6 + b^8 \\
\underline{-a^7b - a^6b^2 - a^5b^3 - a^4b^4 - a^3b^5} \\
a^6b^2 + a^4b^4 + a^3b^5 + a^2b^6 + b^8 \\
\underline{a^6b^2 + a^5b^3 + a^4b^4 + a^3b^5 + a^2b^6} \\
-a^5b^3 + b^8 \\
\underline{-a^5b^3 - a^4b^4 - a^3b^5 - a^2b^6 - ab^7} \\
a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8 \\
\underline{a^4b^4 + a^3b^5 + a^2b^6 + ab^7 + b^8}
\end{array}$$

31.
$$\begin{array}{l}
x^4 - x^3y + x^2y^2 - xy^3 + y^4)\,x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8\,(x^4 + x^3y + x^2y^2 + xy^3 + y^4 \\
\underline{x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4} \\
x^7y + x^5y^3 + x^2y^6 + y^8 \\
\underline{x^7y - x^6y^2 + x^5y^3 - x^4y^4 + x^3y^5} \\
x^6y^2 + x^4y^4 - x^3y^5 + x^2y^6 + y^8 \\
\underline{x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6} \\
x^5y^3 + y^8 \\
\underline{x^5y^3 - x^4y^4 + x^3y^5 - x^2y^6 + xy^7} \\
x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8 \\
\underline{x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8}
\end{array}$$

32.
$$\begin{array}{l}
2x^2 + 3x + 2)\,4x^5 - x^3 + 4x\,(2x^3 - 3x^2 + 2x \\
\underline{4x^5 + 6x^4 + 4x^3} \\
-6x^4 - 5x^3 + 4x \\
\underline{-6x^4 - 9x^3 - 6x^2} \\
4x^3 + 6x^2 + 4x \\
\underline{4x^3 + 6x^2 + 4x}
\end{array}$$

33. $a-3)a^5-243(a^4+3a^3+9a^2+27a+81$

$\underline{a^5-3a^4}$

$3a^4-243$

$\underline{3a^4-9a^3}$

$9a^3-243$

$\underline{9a^3-27a^2}$

$27a^2-243$

$\underline{27a^2-81a}$

$81a-243$

$\underline{81a-243}$

34. $k^3-1)k^{10}-k(k^7+k^4+k$

$\underline{k^{10}-k^7}$

k^7-k

$\underline{k^7-k^4}$

k^4-k

$\underline{k^4-k}$

35. $x+4)x^3-5x^2-46x-40(x^2-9x-10$

$\underline{x^3+4x^2}$

$-9x^2-46x$

$\underline{-9x^2-36x}$

$-10x-40$

$\underline{-10x-40}$

36. $2x-3a)48x^3-76ax^2-64a^2x+105a^3(24x^2-2ax-35a^2$

$\underline{48x^3-72ax^2}$

$-4ax^2-64a^2x$

$\underline{-4ax^2+6a^2x}$

$-70a^2x+105a^3$

$\underline{-70a^2x+105a^3}$

37. $3x^2-4x+5)18x^4-45x^3+82x^2-67x+40(6x^2-7x+8$

$\underline{18x^4-24x^3+30x^2}$

$-21x^3+52x^2-67x$

$\underline{-21x^3+28x^2-35x}$

$24x^2-32x+40$

$\underline{24x^2-32x+40}$

38. $2x - 3a)16x^4 - 72a^2x^2 + 81a^4(8x^3 + 12ax^2 - 18a^2x - 27a^3$

$$\begin{array}{l}
\underline{16x^4 - 24ax^3} \\
\quad 24ax^3 - 72a^2x^2 \\
\quad \underline{24ax^3 - 36a^2x^2} \\
\qquad -36a^2x^2 + 81a^4 \\
\qquad \underline{-36a^2x^2 + 54a^3x} \\
\qquad\quad -54a^3x + 81a^4 \\
\qquad\quad \underline{-54a^3x + 81a^4}
\end{array}$$

39. $3x + 4a)81x^4 - 256a^4(27x^3 - 36ax^2 + 48a^2x - 64a^3$

$$\begin{array}{l}
\underline{81x^4 + 108ax^3} \\
\quad -108ax^3 - 256a^4 \\
\quad \underline{-108ax^3 - 144a^2x^2} \\
\qquad 144a^2x^2 - 256a^4 \\
\qquad \underline{144a^2x^2 + 192a^3x} \\
\qquad\quad -192a^3x - 256a^4 \\
\qquad\quad \underline{-192a^3x - 256a^4}
\end{array}$$

40. $a^2 - b^2)2a^3 + 3a^2b - 2ab^2 - 3b^3(2a + 3b$

$$\begin{array}{l}
\underline{2a^3 - 2ab^2} \\
\quad 3a^2b - 3b^3 \\
\quad \underline{3a^2b - 3b^3}
\end{array}$$

41. $x^2 - a^2)x^3 + 2ax^2 - a^2x - 2a^3(x + 2a$

$$\begin{array}{l}
\underline{x^3 - a^2x} \\
\quad 2ax^2 - 2a^3 \\
\quad \underline{2ax^2 - 2a^3}
\end{array}$$

42. $a^2 + 3b^2)a^4 - a^2b^2 - 12b^4(a^2 - 4b^2$

$$\begin{array}{l}
\underline{a^4 + 3a^2b^2} \\
\quad -4a^2b^2 - 12b^4 \\
\quad \underline{-4a^2b^2 - 12b^4}
\end{array}$$

43. $x^2+3x+y)x^4-9x^2-6xy-y^2(x^2-3x-y$

$$\underline{x^4+3x^3+x^2y}$$
$$-3x^3-x^2y-9x^2-6xy-y^2$$
$$\underline{-3x^3-9x^2-3xy}$$
$$-x^2y-3xy-y^2$$
$$\underline{-x^2y-3xy-y^2}$$

44. $x^2-3xy+2y^2)x^4-6x^3y+9x^2y^2-4y^4(x^2-3xy-2y^2$

$$\underline{x^4-3x^3y+2x^2y^2}$$
$$-3x^3y+7x^2y^2-4y^4$$
$$\underline{-3x^3y+9x^2y^2-6xy^3}$$
$$-2x^2y^2+6xy^3-4y^4$$
$$\underline{-2x^2y^2+6xy^3-4y^4}$$

45. $x-3y)x^4-81y^4(x^3+3x^2y+9xy^2+27y^3$

$$\underline{x^4-3x^3y}$$
$$3x^3y-81y^4$$
$$\underline{3x^3y-9x^2y^2}$$
$$9x^2y^2-81y^4$$
$$\underline{9x^2y^2-27xy^3}$$
$$27xy^3-81y^4$$
$$\underline{27xy^3-81y^4}$$

46. $a-2b)a^4-16b^4(a^3+2a^2b+4ab^2+8b^3$

$$\underline{a^4-2a^3b}$$
$$2a^3b-16b^4$$
$$\underline{2a^3b-4a^2b^2}$$
$$4a^2b^2-16b^4$$
$$\underline{4a^2b^2-8ab^3}$$
$$8ab^3-16b^4$$
$$\underline{8ab^3-16b^4}$$

47. $$\begin{array}{l}
3a+2b)\,81a^4-16b^4\,(27a^3-18a^2b+12ab^2-8b^3 \\
\qquad\qquad \underline{81a^4+54a^3b} \\
\qquad\qquad\qquad -54a^3b-16b^4 \\
\qquad\qquad\qquad \underline{-54a^3b-36a^2b^2} \\
\qquad\qquad\qquad\qquad\qquad 36a^2b^2-16b^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{36a^2b^2+24ab^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad -24ab^3-16b^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-24ab^3-16b^4}
\end{array}$$

48. $$\begin{array}{l}
2x+3y)\,16x^4-81y^4\,(8x^3-12x^2y+18xy^2-27y^3 \\
\qquad\qquad \underline{16x^4+24x^3y} \\
\qquad\qquad\qquad -24x^3y-81y^4 \\
\qquad\qquad\qquad \underline{-24x^3y-36x^2y^2} \\
\qquad\qquad\qquad\qquad\qquad 36x^2y^2-81y^4 \\
\qquad\qquad\qquad\qquad\qquad \underline{36x^2y^2+54xy^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad -54xy^3-81y^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-54xy^3-81y^4}
\end{array}$$

49. $$\begin{array}{l}
a+2b+3c)\,3a^2+8ab+4b^2+10ac+8bc+3c^2\,(3a+2b+c \\
\qquad\qquad \underline{3a^2+6ab+9ac} \\
\qquad\qquad\qquad\qquad 2ab+4b^2+ac+8bc \\
\qquad\qquad\qquad\qquad \underline{2ab+4b^2+6bc} \\
\qquad\qquad\qquad\qquad\qquad\qquad ac+2bc+3c^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac+2bc+3c^2}
\end{array}$$

50. $$\begin{array}{l}
a^2+2ax+4x^2)\,a^4+4a^2x^2+16x^4\,(a^2-2ax+4x^2 \\
\qquad\qquad\qquad \underline{a^4+2a^3x\;+4a^2x^2} \\
\qquad\qquad\qquad\qquad -2a^3x\;+16x^4 \\
\qquad\qquad\qquad\qquad \underline{-2a^3x\;-4a^2x^2-8ax^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad 4a^2x^2+8ax^3+16x^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{4a^2x^2+8ax^3+16x^4}
\end{array}$$

51. $$\begin{array}{l} x^2 - xy + y^2)\,x^4 + x^2y^2 + y^4\,(x^2 + xy + y^2 \\ \underline{x^4 - x^3y + x^2y^2} \\ x^3y + y^4 \\ \underline{x^3y - x^2y^2 + xy^3} \\ x^2y^2 - xy^3 + y^4 \\ \underline{x^2y^2 - xy^3 + y^4} \end{array}$$

52. $$\begin{array}{l} 16x^2 + 4xy + y^2)\,256x^4 + 16x^2y^2 + y^4\,(16x^2 - 4xy + y^2 \\ \underline{256x^4 + 64x^3y + 16x^2y^2} \\ -64x^3y + y^4 \\ \underline{-64x^3y - 16x^2y^2 - 4xy^3} \\ 16x^2y^2 + 4xy^3 + y^4 \\ \underline{16x^2y^2 + 4xy^3 + y^4} \end{array}$$

53. $$\begin{array}{l} x^3 + x - y)\,x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3\,(x^2 + xy - y^2 \\ \underline{x^5 + x^3 - x^2y} \\ x^4y - x^3y^2 + x^2y - 2xy^2 + y^3 \\ \underline{x^4y + x^2y - xy^2} \\ -x^3y^2 - xy^2 + y^3 \\ \underline{-x^3y^2 - xy^2 + y^3} \end{array}$$

54. $$\begin{array}{l} x - a)\,ax^3 + 3a^2x^2 - 2a^3x - 2a^4\,(ax^2 + 4a^2x + 2a^3 \\ \underline{ax^3 - a^2x^2} \\ 4a^2x^2 - 2a^3x \\ \underline{4a^2x^2 - 4a^3x} \\ 2a^3x - 2a^4 \\ \underline{2a^3x - 2a^4} \end{array}$$

55. $$\begin{array}{l} a + x)\,a^2 - x^2\,(a - x \\ \underline{a^2 + ax} \\ -ax - x^2 \\ \underline{-ax - x^2} \end{array}$$

56. $$\begin{array}{l} 2x + 3y + z)\,2x^2 + xy - 3y^2 - 4yz - xz - z^2\,(x - y - z \\ \underline{2x^2 + 3xy + xz} \\ -2xy - 3y^2 - 4yz - 2xz - z^2 \\ \underline{-2xy - 3y^2 - yz} \\ -2xz - 3yz - z^2 \\ \underline{-2xz - 3yz - z^2} \end{array}$$

57.
$$\begin{array}{l}
1+5x+x^2)\,2+9x+14x^3+3x^4\,(2-x+3x^2\\
\qquad\qquad\quad \underline{2+10x+2x^2}\\
\qquad\qquad\qquad \underline{-x-2x^2+14x^3}\\
\qquad\qquad\qquad -x-5x^2-x^3\\
\qquad\qquad\qquad\qquad\quad \overline{3x^2+15x^3+3x^4}\\
\qquad\qquad\qquad\qquad\quad \underline{3x^2+15x^3+3x^4}
\end{array}$$

58.
$$\begin{array}{l}
3-5x+7x^2)\,12-38x+82x^2-112x^3+106x^4-70x^5\,(4-6x+8x^2-10x^3\\
\qquad\qquad\quad \underline{12-20x+28x^2}\\
\qquad\qquad\qquad -18x+54x^2-112x^3\\
\qquad\qquad\qquad \underline{-18x+30x^2-42x^3}\\
\qquad\qquad\qquad\qquad\quad 24x^2-70x^3+106x^4\\
\qquad\qquad\qquad\qquad\quad \underline{24x^2-40x^3+56x^4}\\
\qquad\qquad\qquad\qquad\qquad -30x^3+50x^4-70x^5\\
\qquad\qquad\qquad\qquad\qquad \underline{-30x^3+50x^4-70x^5}
\end{array}$$

59.
$$\begin{array}{l}
x^4-x^3y+x^2y^2-xy^3+y^4)\,x^5+y^5\,(x+y\\
\qquad\qquad\qquad\qquad\quad \underline{x^5-x^4y+x^3y^2-x^2y^3+xy^4}\\
\qquad\qquad\qquad\qquad\qquad x^4y-x^3y^2+x^2y^3-xy^4+y^5\\
\qquad\qquad\qquad\qquad\qquad \underline{x^4y-x^3y^2+x^2y^3-xy^4+y^5}
\end{array}$$

60.
$$\begin{array}{l}
ax+xy+ab+by)\,a^2x^2-a^2b^2-x^2y^2+b^2y^2\,(ax-xy-ab+by\\
\qquad\qquad\quad \underline{a^2x^2+ax^2y+a^2bx+abxy}\\
\qquad\qquad\qquad -ax^2y-a^2bx-abxy-a^2b^2-x^2y^2+b^2y^2\\
\qquad\qquad\qquad \underline{-ax^2y-x^2y^2-abxy-bxy^2}\\
\qquad\qquad\qquad\qquad -a^2bx+bxy^2-a^2b^2+b^2y^2\\
\qquad\qquad\qquad\qquad \underline{-a^2bx-abxy-a^2b^2-ab^2y}\\
\qquad\qquad\qquad\qquad\qquad abxy+bxy^2+ab^2y+b^2y^2\\
\qquad\qquad\qquad\qquad\qquad \underline{abxy+bxy^2+ab^2y+b^2y^2}
\end{array}$$

61. $ax + by)\,abx^2 + a^2xy + b^2xy + aby^2\,(bx + ay$

$$\underline{abx^2 + b^2xy}$$

$$a^2xy + aby^2$$

$$\underline{a^2xy + aby^2}$$

62. $x^2 + ax + b^2)\,x^4 + 2b^2x^2 - a^2x^2 + b^4\,(x^2 - ax + b^2$

$$\underline{x^4 + ax^3 + b^2x^2}$$

$$-ax^3 + b^2x^2 - a^2x^2$$

$$\underline{-ax^3 - a^2x^2 - ab^2x}$$

$$b^2x^2 + ab^2x + b^4$$

$$\underline{b^2x^2 + ab^2x + b^4}$$

XV.

1. $x^2 - ax + c)\,x^4 - (a^2 - b - c)x^2 - (b - c)ax + bc\,(x^2 + ax + b$

$$\underline{x^4 - ax^3 + cx^2}$$

$$ax^3 - (a^2 - b)x^2 - (ab - ac)x + bc$$

$$\underline{ax^3 - a^2x^2 + acx}$$

$$bx^2 - abx + bc$$

$$\underline{bx^2 - abx + bc}$$

2. $y - n)\,y^3 - (l + m + n)y^2 + (lm + ln + mn)y - lmn\,(y^2 - (l + m)y + lm$

$$\underline{y^3 - ny^2}$$

$$-(l + m)y^2 + (lm + ln + mn)y - lmn$$

$$\underline{-(l + m)y^2 + (ln + mn)y}$$

$$lmy - lmn$$

$$\underline{lmy - lmn}$$

3. $$\begin{array}{l} x^3 - mx^2 + nx + r)x^5 - (m-c)x^4 + (n - cm + d)x^3 + (r + cn - dm)x^2 + \\ \qquad\qquad\qquad\qquad\qquad\qquad (cr + dn)x + dr(x^2 + cx + d \\ \qquad\qquad\qquad \underline{x^5 - mx^4 + nx^3 + rx^2} \\ \qquad\qquad\qquad\qquad cx^4 + (d - cm)x^3 + (cn - dm)x^2 \\ \qquad\qquad\qquad\qquad \underline{cx^4 - cmx^3 + cnx^2 + crx\qquad\qquad} \\ \qquad\qquad\qquad\qquad\qquad dx^3 - dmx^2 + dnx + dr \\ \qquad\qquad\qquad\qquad\qquad \underline{dx^3 - dmx^2 + dnx + dr} \end{array}$$

4. $$\begin{array}{l} x^2 + 5x - 4)x^4 + (5 + a)x^3 - (4 - 5a + b)x^2 - (4a + 5b)x + 4b(x^2 + ax - b \\ \qquad\qquad\quad \underline{x^4 + 5x^3 - 4x^2} \\ \qquad\qquad\qquad ax^3 + (5a - b)x^2 - (4a + 5b)x \\ \qquad\qquad\qquad \underline{ax^3 + 5ax^2 - 4ax\qquad\qquad} \\ \qquad\qquad\qquad\quad -bx^2 - 5bx + 4b \\ \qquad\qquad\qquad\quad \underline{-bx^2 - 5bx + 4b} \end{array}$$

5. $$\begin{array}{l} x^2 - (a + c)x + ac)x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 \\ \qquad\qquad\qquad\qquad - (abc + abd + acd + bcd)x + abcd(x^2 - (b + d)x + bd \\ \qquad\qquad\qquad \underline{x^4 - (a + c)x^3 + acx^2} \\ \qquad\qquad\qquad\quad - (b + d)x^3 + (ab + ad + bc + bd + cd)x^2 \\ \qquad\qquad\qquad\quad \underline{- (b + d)x^3 + (ab + ad + bc + cd)x^2 - (abc + acd)x} \\ \qquad\qquad\qquad\qquad\qquad\qquad bdx^2 - (abd + bcd)x + abcd \\ \qquad\qquad\qquad\qquad\qquad\qquad \underline{bdx^2 - (abd + bcd)x + abcd} \end{array}$$

XVIII.

1. $(x^2 - ax) - (bx - ab) = x(x - a) - b(x - a) = (x - b)\ (x - a)$.

2. $(ab + ax) - (bx + x^2) = a(b + x) - x(b + x) = (a - x)\ (b + x)$.

3. $(bc + by) - (cy + y^2) = b(c + y) - y(c + y) = (b - y)\ (c + y)$.

4. $(bm + mn) + (ab + an) = m(b + n) + a(b + n) = (a + m)\ (b + n)$.

5. $(abx^2 - axy) + (bxy - y^2) = ax(bx - y) + y(bx - y) = (ax + y)(bx - y).$

6. $(abx - aby) + (cdx - cdy) = ab(x - y) + cd(x - y) = (ab + cd)(x - y).$

7. $$(cdx^2 + dmxy) - (cnxy + mny^2) = dx(cx + my) - ny(cx + my)$$
$$= (cx + my)(dx - ny).$$

8. $$(abcx - b^2dx) - (acdy - bd^2y) = bx(ac - bd) - dy(ac - bd)$$
$$= (ac - bd)(bx - dy).$$

XXVI.

15. $(x^2 - 2xy + y^2) - z^2 = (x - y)^2 - z^2 = (x - y + z)(x - y - z).$

17. $(a^2 - 2ac + c^2) - (b^2 + 2bd + d^2) = (a - c)^2 - (b + d)^2$, etc.

18. $a^2 - (b^2 - 2bc + c^2) = a^2 - (b - c)^2$, etc.

19. $(x^2 + 2xy + y^2) - z^2 = (x + y)^2 - z^2$, etc.

20. $(a^2 - 2ab + b^2) - (m^2 - 2mn + n^2) = (a - b)^2 - (m - n)^2$, etc.

23. $1 - (a^2 - 2ab + b^2) = 1 - (a - b)^2$, etc.

24. $1 - (x^2 - 2xy + y^2) = 1 - (x - y)^2$, etc.

25. $x^2 - (y^2 + 2yz + z^2) = x^2 - (y + z)^2$, etc.

26. $a^2 - (4b^2 - 12bc + 9c^2) = a^2 - (2b - 3c)^2$, etc.

29. $(a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2$, etc.

30. $(a^2 - 2ac + c^2) - (b^2 - 2bd + d^2) = (a - c)^2 - (b - d)^2$, etc.

31. $3ax(a^2x^2 - 9) = 3ax(ax + 3)(ax - 3).$

XXVII.

11. $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$

12. $x^6 - 1 = (x^3 + 1)(x^3 - 1) = (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1).$

13. $a^6 - 64 = a^6 - 2^6 = (a^3 + 2^3)(a^3 - 2^3)$
$= (a+2)(a^2 - 2a + 4)(a - 2)(a^2 + 2a + 4).$

14. $729 - y^6 = 3^6 - y^6 = (3^3 + y^3)(3^3 - y^3)$
$= (3 + y)(9 - 3y + y^2)(3 - y)(9 + 3y + y^2)$

XXX.

1. $6 + 4 - 5 - 3 = 2.$
2. $6 + 3 - 5 - 4 = 0.$
3. $18 + 12 - 5 - 8 = 17.$
4. $3 \times 11 - 2 \times 1 = 33 - 2 = 31.$
5. $10 \times 2 = 20.$
6. $12 + 3 \times 7 = 12 + 21 = 33.$
7. $(12 + 3)(4 + 3) = 15 \times 7 = 105.$
8. $12 + 12 + 3 = 27.$
9. $(25 + 3) \div (6 - 4) = 28 \div 2 = 14.$
10. $6 \times 5 \times 4 = 120.$
11. $6 \times 5 \times 7 = 210.$
12. $6 \times 3 \times 9 \times 9 = 1458.$
13. $6 \times 5 \times 1 \times 1 = 30.$
14. $\sqrt{5 \times 5} = 5.$
15. $\sqrt{y^2} = y = 3.$
16. $(\sqrt{x})^2 = (2)^2 = 4.$
17. $(\sqrt{x} + b)^2 = (2 + 5)^2 = 49.$
18. $\sqrt{5 \times 5 \times 4} = \sqrt{100} = 10.$

19. $\sqrt{2\times 6\times 4\times 3}=\sqrt{144}=12.$

20. $(36+25+3)\div(4+9+3)=64\div 16=4.$

21. $18+(8-3)^2=18+25=43.$

22. $\{6-(5-3)\}\{6-(4-3)\}=4\times 5=20.$

23. $(6-5-3)^2+(6-4+3)^2=4+25=29.$

24. $3\times 8^3+4\times 10^4=1536+40000$, etc.

25. $3\times 1^2+7^2=3+49=52.$

XXXI.

1. $3\times 3\times 2\times 1-3^3+2^3+1^3=18-27+8+1=0.$

2. $3^3+2^3-5^3+3\times 3\times 2\times 5=27+8-125+90=0.$

3. $(a^2+2ac+c^2)-(a^2+c^2)=2ac.$

4. $x^2+y^2-(x^2-2xy+y^2)=2xy.$

5. $(a^2+2ab+b^2)x-(a^2+2abx+b^2x^2)=(a^2+b^2)x-a^2-b^2x^2.$

6. Multiply $2x-m$ by $x+2m$; the result is $2x^2+3mx-2m^2$.
Multiply $2x+n$ by $x-2n$; the result is $2x^2-3nx-2n^2$.
Then multiply the two results together.

7.
$$\begin{array}{l}
ar+b\,)\,acr^3+(bc+ad)r^2+(bd+ae)r+be\,(\,cr^2+dr+e\\
\qquad\underline{acr^3+bcr^2}\\
\qquad\qquad adr^2+(bd+ae)r\\
\qquad\qquad \underline{adr^2+bdr}\\
\qquad\qquad\qquad aer+be\\
\qquad\qquad\qquad \underline{aer+be}
\end{array}$$

The divisor $=10+1$, or 11; the dividend $=1000+200+20+1$, or 1221; the quotient $=100+10+1$, or 111; and $1221\div 11=111$.

8. Multiply $a+b+c$ by $a+b-c$; the result is $(a+b)^2-c^2$, or $a^2+2ab+b^2-c^2$.

Multiply $c+b-a$ by $c+a-b$; the result is $c^2-(a-b)^2$, or $c^2-a^2+2ab-b^2$.

Then $(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)=(2ab)^2-(a^2+b^2-c^2)^2$
$=4a^2b^2-a^4-b^4-c^4-2a^2b^2+2a^2c^2+2b^2c^2$
$=-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2.$

9. $(a+d)(d+c)-(c+d)(d+a)-(a+c)(d-d)$
$=ad+ac+d^2+cd-cd-ac-d^2-ad-0=0.$

10. $0+(4-16)+\{16-(0+6)\}+\{12-(0+6)\}^2$
$=-12+10+6^2=-2+36=34.$

11. $\{4-3+2-1\}\{4+3-2-1\}=2\times4=8$;
and $16-9-4+1+2\times2=8.$

18. $(ac-ad-bc+bd)-(bc-bd-ac+ad)=2ac-2ad-2bc+2bd.$

19. $ab+ay+bx+xy+x-y+ax-by+ax+ay=$ etc.

20. $(x^2+3x+2)-(x^2-3x+2)=6x$, etc.

21. $ax-by+x-y+x^2-xy+ab-ay-bx+xy$, etc.

22. $(6x^2+20x+16)-(6x^2-28x+16)=48x$, etc.

23. $2mx-3ny+x+y+4mx+4nx-4my-4ny+mx+ny=$ etc.

24. $x^2+y^2+z^2+2xy+2xz+2yz+x^2+y^2+z^2$
$=(x^2+2xy+y^2)+(y^2+2yz+z^2)+(x^2+2xz+z^2)$
$=(x+y)^2+(y+z)^2+(x+z)^2.$

25. $4a^2+6ac-4ab+6ab+9bc-6b^2=4a^2+6ac+2ab+9bc-6b^2.$

26. $\dfrac{ab-cd}{cd+e}=\dfrac{63-15}{15+1}=\dfrac{48}{16}=3.$

$(bc-ad)(bd-ce)=(35-27)(21-5)=8\times16=128.$

$\dfrac{b^2-c^2}{c+d}=\dfrac{49-25}{5+3}=\dfrac{24}{8}=3.$

$d^5-c^3=3^5-5^3=243-125=118.$

27. $0-0+2^3+1^3=8+1=9.$

28. $48+4-8=44.$

29. $(1-2-3)^2+(2-1-3)^2+(3-1-2)^2=(-4)^2+(-2)^2+0=16+4$
$=20.$

30. $(1+2-4)^2+(1-2+4)^2+(2+4-1)^2=(-1)^2+(3)^2+(5)^2$
$=1+9+25=35.$

31. $(-1+2)^2+(2-3)^2+(-1-3)^2=(1)^2+(-1)^2+(-4)^2=1+1+16$
$=18.$

32. Let x and y be the numbers; then $(x^2-y^2)\div(x+y)=x-y$.

33. Let x and y be the numbers; then $(x+y)(x-y)=x^2-y^2$.

34. Let x and $x+1$ be the integers; then $(x+x+1)^2=4x^2+4x+1$
$=4x(x+1)+1.$

35. Let x and $x+2$ be the two even numbers; then $x+1$ is the odd number between them; and $(x+x+2)^2=4x^2+8x+4=4(x+1)^2$.

36. Let x and y be the parts; then $x+y=2$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=2(x-y)$.

37. Let x and y be the parts; then $x+y=50$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=50(x-y)$.

38. Let x and y be the parts; then $x+y=n$;
and $x^2-y^2=(x+y)(x-y)$; therefore $x^2-y^2=n(x-y)$.

39. Let x and $x+1$ be the numbers; then $x(x+1)+x^2+(x+1)^2$
$=x^2+x+x^2+x^2+2x+1=3x^2+3x+1=(x+1)^3-x^3.$

40. Let $x-1$, x, $x+1$ be the three numbers; then $(x-1)^3+x^3+(x+1)^3$
$=x^3-3x^2+3x-1+x^3+x^3+3x^2+3x+1=3x^3+6x=3x(x^2+2).$

XXXII.

1. $7x-5x=11-5$; $2x=6$; $x=3$.

2. $12x-8x=15-7$; $4x=8$; $x=2$.

3. $236x-97x=564-425$; $139x=139$; $x=1$.

4. $5x-3x=7+7$; $2x=14$; $x=7$.

5. $12x-8x=9-1$; $4x=8$; $x=2$.

6. $124x-112x=43-19$; $12x=24$; $x=2$.

7. $5x-2x=27-18$; $3x=9$; $x=3$.

8. $12x-7x=145-125$; $5x=20$; $x=4$.

9. $14x-8x=80-26$; $6x=54$; $x=9$.

10. $-3x-x=-83-133$; $-4x=-216$; $4x=216$; $x=54$.

11. $-3x-5x=-3-13$; $-8x=-16$; $8x=16$; $x=2$.

12. $9x-12x=100-127$; $-3x=-27$; $3x=27$; $x=9$.

13. $-5x+4x=6-15$; $-x=-9$; $x=9$.

14. $3x-7x=6+22$; $-4x=28$; $4x=-28$; $x=-7$.

15. $4x-12x=-16-8$; $-8x=-24$; $8x=24$; $x=3$.

16. $5x-3x+7=4x-6x+35$; $4x=28$; $x=7$.

17. $6x-18+8x+15x-21=10x-4-16x+35$; $35x=70$; $x=2$.

18. $9x-15x+18+30=0$; $-6x=-48$; $x=8$.

19. $12x-45x-15+42-48x+783=0$; $-81x=-810$; $x=10$.

20. $x-28x+77=14x-70-152+19x-61$; $-60x=-360$; $x=6$.

21. $x^2+4x-21=x^2-20x+75$; $24x=96$; $x=4$.

22. $x^2+4x-96=x^2-5x-6$; $9x=90$; $x=10$.

23. $9x-x^2-14+x^2-2x-15-2x+2+12=0$; $5x=15$; $x=3$.

24. $2x^2+3x-35=36-17x+2x^2+229$; $20x=300$; $x=15$.

25. $21-32x+12x^2=12x^2-17x+6$; $-15x=-15$; $x=1$.

26. $14-x-5x^2+5x+30+20-29x+5x^2=45x-76$; $-70x=-140$; $x=2$.

27. $x^2+10x+25-16+8x-x^2=21x$; $-3x=-9$; $x=3$.

28. $5x^2-20x+20+7x^2-42x+63=12x^2-85x+133+42$; $23x=92$; $x=4$.

29. $9x^2-102x+289+16x^2-200x+625-25x^2+290x-841=1$; $-12x=-72$; $x=6$.

30. $x^2-4x-45+x^2+2x-80=2x^2-11x-21-113$; $9x=-9$; $x=-1$.

XXXIII.

1. Let x be the number; then $2x+14=154$; $x=70$.

2. Let x be the number; then $4x+16=188$; $x=43$.

3. Let x be the number; then $x+46=3x$; $x=23$.

4. Let x be the smaller number; then $3x$ is the greater number; and $16-x=30-3x$; $2x=14$; $x=7$; $3x=21$.

5. Let x be the first part; then $x-10$ is the second; $x-18$ the third; $x-24$ the fourth; and $x+x-10+x-18+x-24=92$; $x=36$, etc.

6. Let x be the greater number; then $20-x$ is the smaller number; and $3(20-x)+5x=84$; $x=12$, etc.

7. Let x be the father's age in years; then $80-x$ is the son's age; and $2(80-x)=x+10$; $x=50$, etc.

8. Let x be the age of the eldest; then $x-20$ is the age of the youngest; and $x=3(x-20)$; $x=30$, etc.

9. Let x be the sum in pounds; then $x+24-80=80-x$; $x=68$.

10. Let x be the price of a yard of cloth in shillings; then $2x$ is the price of a yard of silk; and $30x+80x=66\times20$; $x=12$, etc.

11. Let x be the number; then $2x+24-80=100-x$; $x=52$.

12. Let x be A's share in pounds; then B's share is $280-x$; C's, $260-x$; D's, $220-x$; and $x+280-x+260-x+220-x=500$; $x=130$, etc.

13. Let x be the number of children; then there are $2x$ women and $4x$ men; and $x+2x+4x=266$; $x=38$, etc.

14. Let x be B's share in pounds; then A's share is $x-100$, and C's is $x+270$; and $x+x-100+x+270=1520$; $x=450$, etc.

15. Let x be the greater; then $x-8$ is the less; and $4(x-8)=2x+10$; $x=21$; $x-8=13$.

16. Let x be what each had at first in pounds; then $3(x+5)=11(x-5)$; $x=$ £8, 15s.

17. Let x be A's age; then $x-58$ is B's age; and $x-60=50-(x-58)$; $x=84$, etc.

18. Let x be A's age; then $x-34$ is B's age; and $x-50=40-(x-34)$; $x=62$, etc.

19. Let x be the share of a daughter; then $2x$ is the share of a son; and $3x+4x+500$ is the share of the wife; then $3x+4x+3x+4x+500=7500$; $x=500$, etc.

20. Let x be the number of gallons in the vessel at first; then $x+42=7x$; $x=7$; and $42+7=49$, the number of gallons that the vessel held.

21. Let x be the number of pounds A has; then B has $x+10$; C has $x+x+10$; and $x+x+10+x+x+10=76$; $x=14$, etc.

22. Let x be the greater; then $x-14$ is the less; and $x+x-14=48$; $x=31$, etc.

23. Let x be the number of pounds won by A; then $72+x=3(52-x)$; $x=21$.

24. Let x be one of the parts; then $84-x$ is the other; and $3x=4(84-x)$; $x=48$.

25. Let x be one of the parts; then $90-x$ is the other; and $4x=5(90-x)$; $x=50$.

26. Let x be the greater part; then $60-x$ is the less; and $x=60-x+24$; $x=42$.

27. Let x be the greater part; then $84-x$ is the less; and $x-36=84-x$; $x=60$.

28. Let x be one of the parts; then $20-x$ is the other; and $3x+5(20-x)=84$; $x=8$.

29. Let x be B's age; then $2x$ is A's age; and $2x-22=3(x-22)$; $x=44$; $2x=88$.

30. Let x be the number of years; then $30+x=2(6+x)$; $x=18$.

31. Let x be B's age; then $2x$ is A's age; and $2x-20=3(x-20)$; $x=40$.

32. Let x be B's age; then $3x$ is A's age; and $3x+19=2(x+19)$; $x=19$.

33. Let x be the number of years; then $50+x=42+3x$; $x=4$.

34. Let x be the number of guineas; then $x+48$ is the number of half-crowns; and, expressing all the quantities as sixpences, $42x+5(x+48)=100\times 40$; $47x=3760$; $x=80$.

35. Let x be the number of shillings; then $41-x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+5(41-x)=74\times 2$; $-3x=-57$; $x=19$.

36. Let x be the number of shillings; then $300-x$ is the number of fourpenny pieces; and, expressing all the quantities as fourpenny pieces, $3x+300-x=700$; $x=200$.

37. Let x be the number of moidores; then $x+3$ is the number of sovereigns; and $27x+20(x+3)=50\times 20$; $47x=940$; $x=20$.

38. Let x be the number of shillings; then $6x$ is the number of half-crowns; and, expressing all the quantities as sixpences, $2x+30x=1696$; $x=53$.

39. Let x be the number of sovereigns; then $2x$ is the number of shillings, and $3x$ the number of pence; and, expressing all the quantities as pence, $240x+24x+3x=1335$; $x=5$.

XXXV.

1. $a^2-b^2=(a-b)(a+b)$; $a^3-b^3=(a-b)(a^2+ab+b^2)$.

2. $a^4-b^4=(a^2-b^2)(a^2+b^2)$.

3. $a^2-x^2=(a+x)(a-x)$; $(a-x)^2=(a-x)(a-x)$.

4. $a^3+x^3=(a+x)(a^2-ax+x^2)$; $(a+x)^3=(a+x)(a+x)(a+x)$.

5. $9x^2-1=(3x+1)(3x-1)$; $(3x+1)^2=(3x+1)(3x+1)$.

6. $1-25a^2=(1+5a)(1-5a)$; $(1-5a)^2=(1-5a)(1-5a)$.

7. $x^2-y^2=(x+y)(x-y)$; $(x+y)^2=(x+y)(x+y)$;
$x^2+3xy+2y^2=(x+y)(x+2y)$.

8. $x^2-y^2=(x+y)(x-y)$; $x^3-y^3=(x-y)(x^2+xy+y^2)$;
$x^2-7xy+6y^2=(x-y)(x-6y)$.

9. $x^2-1=(x-1)(x+1)$; $x^3-1=(x-1)(x^2+x+1)$;
$x^2+x-2=(x-1)(x+2)$.

10. $1-a^2=(1-a)(1+a)$; $1+a^3=(1+a)(1-a+a^2)$;
$a^2+5a+4=(1+a)(4+a)$.

XXXVI.

1. 6906)10359(1
6906
3453)6906(2
6906

∴ H. C. F. is 3453.

2. 1908)2736(1
1908
828)1908(2
1656
252)828(3
756
72)252(3
216
36)72(2
72

∴ H. C. F. is 36.

3. 49608)169416(3
148824
20592)49608(2
41184
8424)20592(2
16848
3744)8424(2
7488
936)3744(4
3744

∴ H. C. F. is 936.

```
          4. 40115)126025(3
                   120345
                   ------
                     5680)40115(7
                          39760
                          -----
                            355)5680(16
                                355
                                ----
                                2130
   ∴ H. C. F. is 355.           2130
                                ----

5. 1581227)16758766(10
           15812270
           --------
             946496)1581227(1
                    946496
                    ------
                    634731)946496(1
                           634731
                           ------
                           311765)634731(2
                                  623530
                                  ------
                                   11201)311765(27
                                         22402
                                         -----
                                          87745
                                          78407
                                          -----
                                           9338)11201(1
                                                9338
                                                ----
                                                1863)9338(5
                                                     9315
                                                     ----
                                                       23)1863(81
                                                          184
                                                          ---
                                                            23
   ∴ H. C. F. is 23.                                        23
                                                            --
```

6. 35175)236845(6
$\underline{211050}$
25795)35175(1
$\underline{25795}$
9380)25795(2
$\underline{18760}$
7035)9380(1
$\underline{7035}$
2345)7035(3
$\underline{7035}$

∴ H. C. F. is 2345.

XXXVII.

1. $x^2+7x+12)x^2+9x+20(1$
$\underline{x^2+7x+12}$
$2x+8$
Divide by 2 ; $x+4)x^2+7x+12(x+3$
$\underline{x^2+4x}$
$3x+12$
$\underline{3x+12}$

2. $x^2+12x+20)x^2+14x+40(1$
$\underline{x^2+12x+20}$
$2x+20$
Divide by 2 ; $x+10)x^2+12x+20(x+2$
$\underline{x^2+10x}$
$2x+20$
$\underline{2x+20}$

3. $x^2-13x+42)\,x^2-17x+70\,(1$

$$\begin{array}{r}x^2-13x+42\\ \hline -4x+28\end{array}$$

Divide by 4 and change signs ; $x-7)\,x^2-13x+42\,(x-6$

$$\begin{array}{r}x^2-7x\\ \hline -6x+42\\ -6x+42\\ \hline\end{array}$$

4. $x^2+5x-84)\,x^2+21x+108\,(1$

$$\begin{array}{r}x^2+\ 5x-\ 84\\ \hline 16x+192\end{array}$$

Divide by 16 ; $x+12)\,x^2+5x-84\,(x-7$

$$\begin{array}{r}x^2+12x\\ \hline -7x-84\\ -7x-84\\ \hline\end{array}$$

5. $x^2+x-12)\,x^2-2x-3\,(1$

$$\begin{array}{r}x^2+x-12\\ \hline -3x+9\end{array}$$

Change signs and divide by 3 ; $x-3)\,x^2+x-12\,(x+4$

$$\begin{array}{r}x^2-3x\\ \hline 4x-12\\ 4x-12\\ \hline\end{array}$$

6. $x^2+5xy+6y^2)\,x^2+6xy+9y^2\,(1$

$$\begin{array}{r}x^2+5xy+6y^2\\ \hline xy+3y^2\end{array}$$

Divide by y ; $x+3y)\,x^2+5xy+6y^2\,(x+2y$

$$\begin{array}{r}x^2+3xy\\ \hline 2xy+6y^2\\ 2xy+6y^2\\ \hline\end{array}$$

7. $x^2 - 6xy + 8y^2)\,x^2 - 8xy + 16y^2\,(1$

$$\underline{x^2 - 6xy + 8y^2}$$

$$-2xy + 8y^2$$

Divide by $2y$ and change signs ; $x - 4y)\,x^2 - 6xy + 8y^2\,(x - 2y$

$$\underline{x^2 - 4xy}$$

$$-2xy + 8y^2$$

$$\underline{-2xy + 8y^2}$$

8. $x^2 - 13xy - 30y^2)\,x^2 - 18xy + 45y^2\,(1$

$$\underline{x^2 - 13xy - 30y^2}$$

$$-5xy + 75y^2$$

Divide by $5y$ and change signs ; $x - 15y)\,x^2 - 13xy - 30y^2\,(x + 2y$

$$\underline{x^2 - 15xy}$$

$$2xy - 30y^2$$

$$\underline{2xy - 30y^2}$$

9. $x^2 - 2xy + y^2)\,x^3 - y^3\,(x + 2y$

$$\underline{x^3 - 2x^2y + xy^2}$$

$$2x^2y - xy^2 - y^3$$

$$\underline{2x^2y - 4xy^2 + 2y^3}$$

$$3xy^2 - 3y^3$$

Divide by $3y^2$; $x - y)\,x^2 - 2xy + y^2\,(x - y$.

10. $x^3 + y^3)\,x^3 + 3x^2y + 3xy^2 + y^3\,(1$

$$\underline{x^3 \qquad\qquad + y^3}$$

$$3x^2y + 3xy^2$$

Divide by $3xy$; $x + y)\,x^3 + y^3\,(x^2 - xy + y^2$.

11. $x^2 - 2xy + y^2)x^4 - y^4(x^2 + 2xy + 3y^2$

$\underline{x^4 - 2x^3y + x^2y^2}$

$2x^3y - x^2y^2 - y^4$

$\underline{2x^3y - 4x^2y^2 + 2xy^3}$

$3x^2y^2 - 2xy^3 - y^4$

$\underline{3x^2y^2 - 6xy^3 + 3y^4}$

$4xy^3 - 4y^4$

Divide by $4y^3$; $x - y)x^2 - 2xy + y^2(x - y$.

12. $x^3 + y^3)x^5 + y^5(x^2$

$\underline{x^5 + x^2y^3}$

$-x^2y^3 + y^5$

Divide by $-y^3$; $x^2 - y^2)x^3 - y^3(x$

$\underline{x^3 - xy^2}$

$xy^2 - y^3$

Divide by y^2; $x - y)x^2 - y^2(x + y$.

13. $x^2 + 2xy + y^2)x^4 - y^4(x^2 - 2xy + 3y^2$

$\underline{x^4 + 2x^3y + x^2y^2}$

$-2x^3y - x^2y^2 - y^4$

$\underline{-2x^3y - 4x^2y^2 - 2xy^3}$

$3x^2y^2 + 2xy^3 - y^4$

$\underline{3x^2y^2 + 6xy^3 + 3y^4}$

$-4xy^3 - 4y^4$

Divide by $-4y^3$; $x + y)x^2 + 2xy + y^2(x + y$.

14. $a^2 - b^2 + 2bc - c^2)a^2 + 2ab + b^2 - 2ac - 2bc + c^2(1$

$\underline{a^2 - b^2 + 2bc - c^2}$

$2ab + 2b^2 - 2ac - 4bc + 2c^2$

Divide by 2; $ab + b^2 - ac - 2bc + c^2 = (ab - ac) + (b^2 - 2bc + c^2)$

$= a(b - c) + (b - c)^2$.

Divide by $b - c$; $a + b - c)a^2 - b^2 + 2bc - c^2(a - b + c$.

15. **Multiply the second expression by 3 ;**

$$12x^2+7xy+y^2)\,84x^2+9xy-3y^2\,(7$$
$$\underline{84x^2+49xy+7y^2}$$
$$-40xy-10y^2$$

Divide by $-10y$; $4x+y)\,12x^2+7xy+y^2\,(3x+y.$

16. **Multiply the second expression by 2 ;**

$$6x^2+xy-y^2)\,78x^2-44xy+6y^2\,(13$$
$$\underline{78x^2+13xy-13y^2}$$
$$-57xy+19y^2$$

Divide by $-19y$; $3x-y)\,6x^2+xy-y^2\,(2x+y.$

17. **Multiply the second expression by 3 ;**

$$15x^2-8xy+y^2)\,120x^2-9xy-3y^2\,(8$$
$$\underline{120x^2-64xy+8y^2}$$
$$55xy-11y^2$$

Divide by $11y$; $5x-y)\,15x^2-8xy+y^2\,(3x-y.$

18. $x^4+x^3-4x^2+x+1)\,x^5-5x^3+5x^2-1\,(x-1$

$$\underline{x^5+x^4-4x^3+x^2+x}$$
$$-x^4-x^3+4x^2-x-1$$
$$\underline{-x^4-x^3+4x^2-x-1}$$

19. $x^4+4x^2+16)\,x^5+x^4-2x^3+17x^2-10x+20\,(x+1$

$$\underline{x^5+4x^3+16x}$$
$$x^4-6x^3+17x^2-26x+20$$
$$\underline{x^4+4x^2+16}$$
$$-6x^3+13x^2-26x+4$$

Change the signs of the remainder, and multiply the divisor by 6 ;

$$6x^3-13x^2+26x-4)\,6x^4+24x^2+96\,(x$$
$$\underline{6x^4-13x^3+26x^2-4x}$$
$$13x^3-2x^2+4x+96$$

Multiply the remainder by 6, and continue the division;

$$78x^3 - 12x^2 + 24x + 576(13$$
$$\underline{78x^3 - 169x^2 + 338x - 52}$$
$$157x^2 - 314x + 628$$

Divide by 157; $x^2 - 2x + 4)6x^3 - 13x^2 + 26x - 4(6x - 1$

$$\underline{6x^3 - 12x^2 + 24x}$$
$$-x^2 + 2x - 4$$
$$\underline{-x^2 + 2x - 4}$$

20. $x^4 + x^2y^2 + y^4)x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4(1$

$$\underline{x^4 \qquad + x^2y^2 \qquad + y^4}$$
$$2x^3y + 2x^2y^2 + 2xy^3$$

Divide by $2xy$; $x^2 + xy + y^2)x^4 + x^2y^2 + y^4(x^2 - xy + y^2$.

21. $x^6 - 6x^4 + 9x^2 - 4)x^6 + x^5 - 2x^4 + 3x^2 - x - 2(1$

$$\underline{x^6 \quad - 6x^4 + 9x^2 \quad - 4}$$
$$x^5 + 4x^4 - 6x^2 - x + 2$$
$$x^5 + 4x^4 - 6x^2 - x + 2)x^6 - 6x^4 + 9x^2 - 4(x - 4$$
$$\underline{x^6 + 4x^5 - 6x^3 - x^2 + 2x}$$
$$-4x^5 - 6x^4 + 6x^3 + 10x^2 - 2x - 4$$
$$\underline{-4x^5 - 16x^4 + 24x^2 + 4x - 8}$$
$$10x^4 + 6x^3 - 14x^2 - 6x + 4$$

Divide by 2, and multiply the divisor by 5;

$$5x^4 + 3x^3 - 7x^2 - 3x + 2)5x^5 + 20x^4 - 30x^2 - 5x + 10(x$$
$$\underline{5x^5 + 3x^4 - 7x^3 - 3x^2 + 2x}$$
$$17x^4 + 7x^3 - 27x^2 - 7x + 10$$

Multiply by 5, and continue the division;

$$85x^4 + 35x^3 - 135x^2 - 35x + 50(17$$
$$\underline{85x^4 + 51x^3 - 119x^2 - 51x + 34}$$
$$-16x^3 - 16x^2 + 16x + 16$$

Divide by -16; $x^3+x^2-x-1)5x^4+3x^3-7x^2-3x+2(5x-2$

$$\begin{array}{r} 5x^4+5x^3-5x^2-5x \\ \hline -2x^3-2x^2+2x+2 \\ -2x^3-2x^2+2x+2 \\ \hline \end{array}$$

22. Multiply the first expression by 2;

$6a^3+19a^2b+8ab^2-5b^3)30a^4+20a^3b+8a^2b^2+12ab^3-6b^4(5a$

$$\begin{array}{r} 30a^4+95a^3b+40a^2b^2-25ab^3 \\ \hline -75a^3b-32a^2b^2+37ab^3-6b^4 \end{array}$$

Change the signs, multiply by 2, divide by b, and continue the division;

$6a^3+19a^2b+8ab^2-5b^3)150a^3+64a^2b-74ab^2+12b^3(25$

$$\begin{array}{r} 150a^3+475a^2b+200ab^2-125b^3 \\ \hline -411a^2b-274ab^2+137b^3 \end{array}$$

Divide by $-137b$; $3a^2+2ab-b^2)6a^3+19a^2b+8ab^2-5b^3(2a+5b$.

23. Multiply the second expression by 5;

$15x^3-14x^2y+24xy^2-7y^3)135x^3+165x^2y-100xy^2+10y^3(9$

$$\begin{array}{r} 135x^3-126x^2y+216xy^2-63y^3 \\ \hline 291x^2y-316xy^2+73y^3 \end{array}$$

Divide by y; and multiply the divisor by 97;

$291x^2-316xy+73y^2)1455x^3-1358x^2y+2328xy^2-679y^3(5x$

$$\begin{array}{r} 1455x^3-1580x^2y+365xy^2 \\ \hline 222x^2y+1963xy^2-679y^3 \end{array}$$

Divide by y; multiply by 97, and continue the division;

$$\begin{array}{r} 21534x^2+190411xy-65863y^2(74 \\ 21534x^2-\ 23384xy\ +\ 5402y^2 \\ \hline 213795xy-71265y^2 \end{array}$$

Divide by $71265y$; $3x-y)291x^2-316xy+73y^2(97x-73y$.

24. Multiply the second expression by 7;

$$21x^2-83xy-27x+22y^2+99y)\,84x^2-245xy-42x-231y^2+154y\,(4$$
$$\underline{84x^2-332xy-108x+88y^2+396y}$$
$$87xy+66x-319y^2-242y$$

The remainder $=87xy-319y^2+66x-242y$
$=29y(3x-11y)+22(3x-11y)$
$=(29y+22)(3x-11y)$

Rejecting $29y+22$, which is clearly not a factor of the divisor;

$$3x-11y)\,21x^2-83xy-27x+22y^2+99y\,(7x-2y-9.$$

25. $3a^3-12a^2-a^2b+10ab-2b^2=(3a^3-a^2b)-(12a^2-10ab+2b^2)$
$=a^2(3a-b)-2(6a^2-5ab+b^2)$
$=a^2(3a-b)-2(3a-b)(2a-b)$
$=(3a-b)(a^2-4a+2b)$

Rejecting $a^2-4a+2b$, which is clearly not a factor of the second expression, we find the H. C. F. to be $3a-b$.

26. $60a^2-75ax+15x^2=15(4a^2-5ax+x^2)=3\times 5(a-x)(4a-x)$

Of these four factors, 5 and $4a-x$ are clearly not factors of the first expression: but 3 is a factor of it: divide it by 3, and divide the result by $a-x$.

$$a-x)\,6a^3-6a^2x+2ax^2-2x^3\,(6a^2+2x^2$$

∴ H. C. F. is $3(a-x)$.

27. Divide the first expression by x, and multiply the result by 2;

$$6x^2-x-2)\,42x^2-52x+16\,(7$$
$$\underline{42x^2-7x-14}$$
$$-45x+30$$

Divide by -15; $3x-2)\,6x^2-x-2\,(2x+1.$

28. $3x^3-15ax^2+a^2x-5a^3)\,6x^4+29a^2x^2+9a^4\,(2x+10a$

$$\underline{6x^4-30ax^3+2a^2x^2-10a^3x}$$
$$30ax^3+27a^2x^2+10a^3x+9a^4$$
$$\underline{30ax^3-150a^2x^2+10a^3x-50a^4}$$
$$177a^2x^2 \qquad +59a^4$$

Divide by $59a^2$; $3x^2+a^2)\,3x^3-15ax^2+a^2x-5a^3\,(x-5a.$

29. $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$

$$x^8 + x^6y^2 + x^2y + y^3 = x^6(x^2 + y^2) + y(x^2 + y^2)$$

$$\therefore x^2 + y^2 \text{ is the H. C. F.}$$

30. Divide the first expression by 2;

$$x^3 + x^2 + 7x + 39)x^3 + 5x^2 + 7x + 3(1$$
$$\underline{x^3 + x^2 + 7x + 39}$$
$$4x^2 - 36$$

Now $4x^2 - 36 = 4(x^2 - 9) = 4(x + 3)(x - 3)$; and rejecting 4 and $x - 3$, we have

$$x + 3)x^3 + x^2 + 7x + 39(x^2 - 2x + 13.$$

31. $45a^3x + 3a^2x^2 - 9ax^3 + 6x^4 = x(45a^3 + 3a^2x - 9ax^2 + 6x^3)$

$18a^2x - 8x^3 = 2x(9a^2 - 4x^2)$

Reserving the common factor x;

$$9a^2 - 4x^2)45a^3 + 3a^2x - 9ax^2 + 6x^3(5a$$
$$\underline{45a^3 - 20ax^2}$$
$$3a^2x + 11ax^2 + 6x^3$$

Divide by x, multiply by 3, and proceed with the division;

$$9a^2 + 33ax + 18x^2(1$$
$$\underline{9a^2 - 4x^2}$$
$$33ax + 22x^2$$

Divide by $11x$; $3a + 2x)9a^2 - 4x^2(3a - 2x$

$\therefore$ the H. C. F. is $(3a + 2x)x$.

XXXVIII.

1. $x^2 + 5x + 6 = (x + 2)(x + 3)$; $x^2 + 7x + 10 = (x + 2)(x + 5)$; $x^2 + 12x + 20 = (x + 2)(x + 10)$.

2. $x^3 + 4x^2 - 5)x^3 - 3x + 2(1$

$$\underline{x^3 + 4x^2 - 5}$$
$$-4x^2 - 3x + 7$$

Change signs and multiply the divisor by 4 ;

$$4x^2+3x-7)\,4x^3+16x^2-20\,(x$$
$$\underline{4x^3+3x^2-7x}$$
$$13x^2+7x-20$$

Multiply by 4, and continue the division ;

$$52x^2+28x-80\,(13$$
$$\underline{52x^2+39x-91}$$
$$-11x+11$$

Change signs and divide by 11 ;

$$x-1)\,4x^2+3x-7\,(4x+7$$

Hence $x-1$ is the H. C. F. of the first two expressions ; and since $x-1$ divides x^3+4x^2-8x+3 exactly, it is the H. C. F. required.

3. $2x^2+x-1=(2x-1)(x+1)$; $x^2+5x+4=(x+4)(x+1)$;
$x^3+1=(x+1)(x^2-x+1)$.

4. $y^3-y^2-y+1=(y^3-y^2)-(y-1)=y^2(y-1)-(y-1)=(y^2-1)(y-1)$
$3y^2-2y-1=(3y+1)(y-1)$
$y^3-y^2+y-1=y^2(y-1)+(y-1)=(y^2+1)(y-1)$.

5. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$
$$\underline{x^3-4x^2+9x-10}$$
$$6x^2-12x+30$$

Divide by 6 ; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$
$$\underline{x^3-2x^2+5x}$$
$$-2x^2+4x-10$$
$$\underline{-2x^2+4x-10}$$

Hence x^2-2x+5 is the H. C. F. of the first two expressions, and as it also divides $x^3+5x^2-9x+35$ exactly, it is the H. C. F. required.

6. $x^3 - 7x^2 + 16x - 12)\, 3x^3 - 14x^2 + 16x\, (3$

$\underline{3x^3 - 21x^2 + 48x - 36}$

$7x^2 - 32x + 36$

Multiply the divisor by 7 ;

$7x^2 - 32x + 36)\, 7x^3 - 49x^2 + 112x - 84\, (x$

$\underline{7x^3 - 32x^2 + 36x}$

$-17x^2 + 76x - 84$

Multiply by -7, and continue the division ;

$119x^2 - 532x + 588\, (17$

$\underline{119x^2 - 544x + 612}$

$12x - 24$

Divide by 12 ; $x - 2)\, 7x^2 - 32x + 36\, (7x - 18$

Hence $x - 2$, which also divides $5x^3 - 10x^2 + 7x - 14$, is the H. C. F.

7. $y^3 - 5y^2 + 11y - 15)\, y^3 - y^2 + 3y + 5\, (1$

$\underline{y^3 - 5y^2 + 11y - 15}$

$4y^2 - 8y + 20$

Divide by 4 ; $y^2 - 2y + 5)\, y^3 - 5y^2 + 11y - 15\, (y - 3$

$\underline{y^3 - 2y^2 + 5y}$

$-3y^2 + 6y - 15$

$\underline{-3y^2 + 6y - 15}$

Hence as $y^2 - 2y + 5$ divides $2y^3 - 7y^2 + 16y - 15$ exactly, it is the H. C. F.

XXXIX.

11. $\frac{a^2}{a^2 + ab} = \frac{a^2}{a(a + b)} = \frac{a}{a + b}$.

12. $\frac{14m^2x}{21m^3p - 7mx} = \frac{14m^2x}{7m(3m^2p - x)} = \frac{2mx}{3m^2p - x}$.

13. $\frac{xy}{3xy^2 - 5x^2yz} = \frac{xy}{xy(3y - 5xz)} = \frac{1}{3y - 5xz}$.

14. $\dfrac{4ax+2x^2}{8ax^3-2x^2}=\dfrac{2x(2a+x)}{2x(4ax^2-x)}=\dfrac{2a+x}{4ax^2-x}.$

15. $\dfrac{ay+y^2}{abc+bcy}=\dfrac{y(a+y)}{bc(a+y)}=\dfrac{y}{bc}.$

16. $\dfrac{4a^2x+6a^2y}{8x^2-18y^2}=\dfrac{2a^2(2x+3y)}{2(2x+3y)\,(2x-3y)}=\dfrac{a^2}{2x-3y}.$

17. $\dfrac{12ab^2-6ab}{8b^2c-2c}=\dfrac{6ab(2b-1)}{2c(2b+1)\,(2b-1)}=\dfrac{3ab}{2bc+c}.$

18. $\dfrac{c^2-4a^2}{c^2+4ac+4a^2}=\dfrac{(c+2a)\,(c-2a)}{(c+2a)\,(c+2a)}=\dfrac{c-2a}{c+2a}.$

19. $\dfrac{3x^4+3x^2y^2}{5x^4+5x^2y^2}=\dfrac{3x^2(x^2+y^2)}{5x^2(x^2+y^2)}=\dfrac{3}{5}.$

20. $\dfrac{10x-10y}{4x^2-8xy+4y^2}=\dfrac{10(x-y)}{4(x-y)\,(x-y)}=\dfrac{5}{2x-2y}.$

21. $\dfrac{ax+by}{7a^2x^2-7b^2y^2}=\dfrac{ax+by}{7(ax+by)\,(ax-by)}=\dfrac{1}{7ax-7by}.$

22. $\dfrac{6ab+8cd}{27a^2b^2x-48c^2d^2x}=\dfrac{2(3ab+4cd)}{3x(3ab+4cd)(3ab-4cd)}=\dfrac{2}{9abx-12cdx}.$

23. $\dfrac{xy-xyz}{2az-2az^2}=\dfrac{xy(1-z)}{2az(1-z)}=\dfrac{xy}{2az}.$

24. $\dfrac{7ab^3x^8-7ab^3y^2}{14a^3bcx^8-14a^3bcy^2}=\dfrac{7ab^3(x^8-y^2)}{14a^3bc(x^8-y^2)}=\dfrac{b^2}{2a^2c}.$

25. $\dfrac{5x^9+45dx^2}{10cx^9+90cdx^2}=\dfrac{5x^2(x^7+9d)}{10cx^2(x^7+9d)}=\dfrac{1}{2c}.$

26. $\dfrac{10a^2+20ab+10b^2}{5a^3+5a^2b}=\dfrac{10(a+b)\,(a+b)}{5a^2(a+b)}=\dfrac{2a+2b}{a^2}.$

27. $\dfrac{4x^2-8xy+4y^2}{48(x-y)^2}=\dfrac{4(x-y)^2}{48(x-y)^2}=\dfrac{1}{12}.$

28. $\dfrac{3mx+5nx^2}{3my+5nxy}=\dfrac{x(3m+5nx)}{y(3m+5nx)}=\dfrac{x}{y}.$

XL.

1. $\frac{a^2+7a+10}{a^2+5a+6}=\frac{(a+5)(a+2)}{(a+3)(a+2)}=\frac{a+5}{a+3}.$

2. $\frac{x^2-9x+20}{x^2-7x+12}=\frac{(x-4)(x-5)}{(x-4)(x-3)}=\frac{x-5}{x-3}.$

3. $\frac{x^2-2x-3}{x^2-10x+21}=\frac{(x-3)(x+1)}{(x-3)(x-7)}=\frac{x+1}{x-7}.$

4. $\frac{x^2-18xy+45y^2}{x^2-8xy-105y^2}=\frac{(x-3y)(x-15y)}{(x+7y)(x-15y)}=\frac{x-3y}{x+7y}.$

5. $\frac{x^4+x^2+1}{x^2+x+1}=\frac{(x^2+x+1)(x^2-x+1)}{x^2+x+1}=x^2-x+1.$

6. $\frac{x^6+2x^3y^3+y^6}{x^6-y^6}=\frac{(x^3+y^3)(x^3+y^3)}{(x^3+y^3)(x^3-y^3)}=\frac{x^3+y^3}{x^3-y^3}.$

7. $x^3-4x^2+9x-10)\,x^3+2x^2-3x+20\,(1$

$$\begin{array}{r} x^3-4x^2+9x-10 \\ \hline 6x^2-12x+30 \end{array}$$

Divide by 6; $x^2-2x+5)\,x^3-4x^2+9x-10\,(x-2$

Hence x^2-2x+5 is the H. C. F.

8. $x^3-5x^2+11x-15)\,x^3-x^2+3x+5\,(1$

$$\begin{array}{r} x^3-5x^2+11x-15 \\ \hline 4x^2-8x+20 \end{array}$$

Divide by 4; $x^2-2x+5)\,x^3-5x^2+11x-15\,(x-3$

Hence x^2-2x+5 is the H. C. F.

9. $x^3-8x^2+21x-18)\,3x^3-16x^2+21x\,(3$

$$\begin{array}{r} 3x^3-24x^2+63x-54 \\ \hline 8x^2-42x+54 \end{array}$$

Divide by 2; $4x^2-21x+27$

Multiply divisor by 4; $4x^2-21x+27)\,4x^3-32x^2+84x-72\,(x$

$$\begin{array}{r}4x^3-21x^2+27x\\ \hline -11x^2+57x-72\end{array}$$

Multiply by -4, and continue the division;

$$\begin{array}{r}4x^2-21x+27)\,44x^2-228x+288\,(11\\ 44x^2-231x+297\\ \hline 3x-9\end{array}$$

Divide by 3; $x-3)\,4x^2-21x+27\,(4x-9$

Hence $x-3$ **is the H. C. F.**

10. **The H. C. F. is** $x-2$, **see the work in xxxviii. 6.**

11. $x^4-x^3y-xy^3-y^4)\,x^4+x^3y+xy^3-y^4\,(1$

$$\begin{array}{r}x^4-x^3y-xy^3-y^4\\ \hline 2x^3y+2xy^3\end{array}$$

Divide by $2xy$; $x^2+y^2)\,x^4-x^3y-xy^3-y^4\,(x^2-xy-y^2.$

12. $a^3-3a+2)\,a^3+4a^2-5\,(1$

$$\begin{array}{r}a^3-3a+2\\ \hline 4a^2+3a-7\end{array}$$

Multiply divisor by 4; $4a^2+3a-7)\,4a^3-12a+8\,(a$

$$\begin{array}{r}4a^3+3a^2-7a\\ \hline -3a^2-5a+8\end{array}$$

Multiply by -4 and continue the division;

$$\begin{array}{r}12a^2+20a-32\,(3\\ 12a^2+9a-21\\ \hline 11a-11\end{array}$$

Divide by 11; $a-1)\,4a^2+3a-7\,(4a+7.$

13. $b^3-6b+5)\,b^3+4b^2-5b\,(1$

$$\begin{array}{r}b^3-6b+5\\ \hline 4b^2+b-5\end{array}$$

Multiply divisor by 4; $$\begin{array}{r} 4b^2+b-5)\,4b^3-24b+20\,(b \\ 4b^3+b^2-5b \\ \hline -b^2-19b+20 \end{array}$$

Change signs; $$\begin{array}{r} b^2+19b-20)\,4b^2+b-5\,(4 \\ 4b^2+76b-80 \\ \hline -75b+75 \end{array}$$

Hence the H. C. F. is found to be $b-1$.

14. $$\begin{array}{r} m^3-7m+6)\,m^3+3m^2-4m\,(1 \\ m^3-7m+6 \\ \hline 3m^2+3m-6 \end{array}$$

Divide by 3; $$\begin{array}{r} m^2+m-2)\,m^3-7m+6\,(m-1 \\ m^3+m^2-2m \\ \hline -m^2-5m+6 \\ -m^2-\ m\ +2 \\ \hline -4m+4 \end{array}$$

Hence the H. C. F. is found to be $m-1$.

15. $$\begin{array}{r} a^3+1)\,a^3+2a^2+2a+1\,(1 \\ a^3 \qquad\qquad +1 \\ \hline 2a^2+2a \end{array}$$

Divide by $2a$, and H. C. F. is found to be $a+1$.

16. $$\frac{3ax^2-13ax+14a}{7x^3-17x^2+6x}=\frac{a(3x^2-13x+14)}{x(7x^2-17x+6)}$$

$$\begin{array}{r} 3x^2-13x+14)\,21x^2-51x+18\,(7 \\ 21x^2-91x+98 \\ \hline 40x-80 \end{array}$$

Whence H. C. F. of $3x^2-13x+14$ and $7x^2-17x+6$ is found to be $x-2$.

17. $$\frac{14x^2-34x+12}{9ax^2-39ax+42a}=\frac{2(7x^2-17x+6)}{3a(3x^2-13x+14)};$$

and as in the preceding question, the H. C. F. is found to be $x-2$.

18. $$\frac{10a-24a^2+14a^3}{15-24a+3a^2+6a^3}=\frac{2a(5-12a+7a^2)}{3(5-8a+a^2+2a^3)}$$

$$5-12a+7a^2)\,5-8a+a^2+2a^3\,(1$$
$$\underline{5-12a+7a^2}$$
$$4a-6a^2+2a^3$$

Divide by $2a$; $2-3a+a^2=(1-a)(1-2a)$

Hence the H. C. F. is found to be $1-a$.

19. $$\frac{2ab^3+ab^2-8ab+5a}{7b^3-12b^2+5b}=\frac{a(2b^3+b^2-8b+5)}{b(7b^2-12b+5)}$$

$$7b^2-12b+5)\,14b^3+7b^2-56b+35\,(2b$$
$$\underline{14b^3-24b^2+10b}$$
$$31b^2-66b+35$$
$$\underline{\qquad 7 \qquad}$$
$$217b^2-462b+245\,(31$$
$$\underline{217b^2-372b+155}$$
$$-90b+90$$

Hence the H. C. F. is found to be $b-1$.

20. $$a^3-4a^2+6a-4)\,a^3-3a^2+3a-2\,(1$$
$$\underline{a^3-4a^2+6a-4}$$
$$a^2-3a+2)\,a^3-4a^2+6a-4\,(a-1$$
$$\underline{a^3-3a^2+2a}$$
$$-a^2+4a-4$$
$$\underline{-a^2+3a-2}$$
$$a-2$$

Hence we find $a-2$ to be the H. C. F.

21. $$\frac{3x^2+2x-1}{x^3+x^2-x-1}=\frac{(3x-1)(x+1)}{(x^2-1)(x+1)}=\frac{3x-1}{x^2-1}.$$

22. $$\frac{a^2-a-20}{a^2+a-12}=\frac{(a-5)(a+4)}{(a-3)(a+4)}=\frac{a-5}{a-3}.$$

23. $x^3 - 3x^2 + 4x - 2)\,x^3 - x^2 - 2x + 2\,(1$
$\underline{x^3 - 3x^2 + 4x - 2}$
$2x^2 - 6x + 4$

Divide by 2; $x^2 - 3x + 2)\,x^3 - 3x^2 + 4x - 2\,(x$
$\underline{x^3 - 3x^2 + 2x}$
$2x - 2$

Hence we find $x - 1$ to be the H. C. F.

24. $$\frac{x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + z^2 - 2yz + y^2 + x^2 - 2xz + z^2 + y^2 - 2xy + x^2}{x^2 + y^2 + z^2}$$
$$= \frac{3x^2 + 3y^2 + 3z^2}{x^2 + y^2 + z^2} = 3.$$

25. Multiply the numerator by 7;
$7x^3 - 19x^2 + 17x - 5)\,14x^4 - 7x^3 - 63x^2 + 91x - 35\,(2x$
$\underline{14x^3 - 38x^3 + 34x^2 - 10x}$
$31x^3 - 97x^2 + 101x - 35$
$\underline{7}$
$217x^3 - 679x^2 + 707x - 245\,(31$
$\underline{217x^3 - 589x^2 + 527x - 155}$
$-90x^2 + 180x - 90$

Divide by -90; $x^2 - 2x + 1)\,7x^3 - 19x^2 + 17x - 5\,(7x - 5$
$\underline{7x^3 - 14x^2 + 7x}$
$-5x^2 + 10x - 5$
$\underline{-5x^2 + 10x - 5}$

Hence $x^2 - 2x + 1$ is the H. C. F.

26. $8x^4 - 30x^3 + 31x^2 - 12)\,16x^4 - 53x^2 + 45x + 6\,(2$
$\underline{16x^4 - 60x^3 + 62x^2 - 24}$
$60x^3 - 115x^2 + 45x + 30$

Divide by 5, and multiply the divisor by 3;

$$\begin{array}{r}
12x^3-23x^2+9x+6)\,24x^4-90x^3+93x^2-36\,(2x \\
12x^4-46x^3+18x^2+12x \\ \hline
-44x^3+75x^2-12x-36
\end{array}$$

Multiply by -3, and continue the division;

$$\begin{array}{r}
132x^3-225x^2+36x+108\,(11 \\
132x^3-253x^2+99x+\ \ 66 \\ \hline
28x^2-63x+\ \ 42
\end{array}$$

Divide by 7; $4x^2-9x+6)\,12x^3-23x^2+9x+6\,(3x+1$

Hence $4x^2-9x+6$ is the H. C. F.

27. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}=\dfrac{(2x-3a)(2x-3a)}{(2x-3a)(4x^2+6ax+9a^2)}$.

28. $$\begin{array}{r}
6x^3-23x^2+16x-3)\,6x^3-17x^2+11x-2\,(1 \\
6x^3-23x^2+16x-3 \\ \hline
6x^2-\ \ 5x+1
\end{array}$$

$6x^2-5x+1)\,6x^3-23x^2+16x-3\,(x-3$

Hence $6x^2-5x+1$ is the H. C. F.

29. $$\begin{array}{r}
x^3-6x^2+11x-6)\,x^3-2x^2-\ \ x\ \ +2\,(1 \\
x^3-6x^2+11x-6 \\ \hline
4x^2-12x+8
\end{array}$$

Divide by 4; $x^2-3x+2)\,x^3-6x^2+11x-6\,(x-3$

Hence x^2-3x+2 is the H. C. F.

30. $$\begin{array}{r}
m^3+m^2+m-3)\,m^3+3m^2+5m+3\,(1 \\
m^3+\ \ m^2+\ \ m-3 \\ \hline
2m^2+4m+6
\end{array}$$

Divide by 2; $m^2+2m+3)\,m^3+m^2+m-3\,(m-1$

Hence m^2+2m+3 is the H. C. F.

31. $\dfrac{x^5+5x^4-x^2-5x}{x^4+3x^3-x-3}=\dfrac{x^4(x+5)-x(x+5)}{x^3(x+3)-(x+3)}=\dfrac{(x+5)(x^3-1)x}{(x+3)(x^3-1)}=\dfrac{x^2+5x}{x+3}$.

32. $\frac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}=\frac{a^2-(b^2+2bc+c^2)}{(a^2+2ab+b^2)-c^2}=\frac{a^2-(b+c)^2}{(a+b)^2-c^2}$

$=\frac{(a+b+c)(a-b-c)}{(a+b+c)(a+b-c)}=\frac{a-b-c}{a+b-c}.$

33. Multiply the numerator by 3;

$$\begin{array}{r}9a^2+3ab-2b^2)\,45a^2+\ \ 3ab-\ \ 6b^2(5\\ \underline{45a^2+15ab-10b^2}\\ -12ab+\ \ 4b^2\end{array}$$

Divide by $-4b$; $3a-b)\,9a^2+3ab-2b^2(3a+2b$

Hence $3a-b$ is the H. C. F.

34. $x^2-7x+10=(x-5)(x-2)$

$x-5$ is clearly not a factor of the denominator;

$x-2)\,2x^2-x-6(2x+3$

Hence $x-2$ is the H. C. F.

35. $$\begin{array}{r}x^3+3x^2+4x+12)\,x^3+4x^2+4x+3\,(1\\ \underline{x^3+3x^2+4x+12}\\ x^2-9\end{array}$$

Divide by $x-3$; $x+3)\,x^3+3x^2+4x+12\,(x^2+4$

Hence $x+3$ is the H. C. F.

36. $$\begin{array}{r}2x^3-x-1)\,2x^4-2x^2-4x+4)\,x\\ \underline{2x^4-\ \ x^2-x}\\ -\ \ x^2-3x+4\end{array}$$

Change signs; $x^2+3x-4=(x+4)(x-1)$

Divide by $x+4$; $x-1)\,2x^3-x-1\,(2x^2+2x+1$

Hence $x-1$ is the H. C. F.

37. $$\begin{array}{r}3x^2-4x-15)\,3x^3-6x^2-45x+108\,(x\\ \underline{3x^3-4x^2-15x}\\ -2x^2-30x+108\end{array}$$

Divide by -2; $x^2+15x-54=(x+18)(x-3)$

Divide by $x+18$; $x-3)\,3x^2-4x-15\,(3x+5$

Hence $x-3$ is the H. C. F.

38. $3x^3+x^2-5x+21)\,6x^3+29x^2+26x-21\,(2$

$\underline{6x^3+\;2x^2-10x+42}$

$27x^2+36x-63$

Divide by 9; $3x^2+4x-7)\,3x^3+\;x^2-5x+21\,(x-1$

$\underline{3x^3+4x^2-7x}$

$-3x^2+2x+21$

$\underline{-3x^2-4x+\;7}$

$6x+14$

Divide by 2; $3x+7)\,3x^2+4x-7\,(x-1$

Hence $3x+7$ is the H. C. F.

39. $4x^3-3x^2-8x-1)\,4x^4-4x^3-16x^2-4x+4\,(x$

$\underline{4x^4-3x^3-\;8x^2-x}$

$-\;x^3-\;8x^2-3x+4$

Change signs; $x^3+8x^2+3x-4)\,4x^3-\;3x^2-\;8x-\;1\,(4$

$\underline{4x^3+32x^2+12x-16}$

$-35x^2-20x+15$

Divide by -5; $7x^2+4x-3=(7x-3)(x+1)$

Divide by $7x-3$; $x+1)\,x^3+8x^2+3x-4\,(x^2+7x-4$

Hence $x+1$ is the H. C. F.

40. $a^3-7a^2+16a-12)\,3a^3-14a^2+16a\,(3$

$\underline{3a^3-21a^2+48a-36}$

$7a^2-32a+36$

$7a^2-32a+36)\,7a^3-49a^2+112a-84\,(a$

$\underline{7a^3-32a^2+\;36a}$

$-17a^2+\;76a-84$

$\underline{7}$

$-119a^2+532a-588\,(-17$

$\underline{-119a^2+544a-612}$

$-12a+24$

Hence we find $a-2$ to be the H. C. F.

XLII.

1. $\dfrac{a-b}{a^2+ab}\times\dfrac{a^2-b^2}{a^2-ab}=\dfrac{(a-b)(a+b)(a-b)}{a(a+b)a(a-b)}=$ etc.

2. $\dfrac{x^2+4x}{x^2-3x}\times\dfrac{4x^2-12x}{3x^2+12x}=\dfrac{x(x+4)4x(x-3)}{x(x-3)3x(x+4)}=$ etc.

3. $\dfrac{x^2+3x+2}{x^2-5x+6}\times\dfrac{x^2-7x+12}{x^2+x}=\dfrac{(x+2)(x+1)(x-3)(x-4)}{(x-2)(x-3)x(x+1)}=$ etc.

4. $\dfrac{x^2+x-2}{x^2-7x}\times\dfrac{x^2-13x+42}{x^2+2x}=\dfrac{(x+2)(x-1)(x-7)(x-6)}{x(x-7)x(x+2)}=$ etc.

5. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}=\dfrac{(x-6)(x-5)x(x-3)}{(x-3)(x-3)x(x-5)}=$ etc.

6. $\dfrac{x^2-4}{x^2+5x}\times\dfrac{x^2-25}{x^2+2x}=\dfrac{(x+2)(x-2)(x+5)(x-5)}{x(x+5)x(x+2)}=$ etc.

7. $\dfrac{(a-3)(a-1)}{(a-4)(a-1)}\times\dfrac{(a-5)(a-4)}{(a-7)(a-3)}\times\dfrac{a(a-7)}{a(a-5)}=1.$

8. $\dfrac{(b-6)(b-1)}{(b+4)(b-1)}\times\dfrac{(b+4)(b+6)}{(b-8)(b-6)}\times\dfrac{b^2(b-8)}{b(b+6)}=b.$

9. $\dfrac{(x+y)(x-y)}{(x-y)(x-2y)}\times\dfrac{y(x-2y)}{x(x+y)}\times\dfrac{x(x-y)}{(x-y)(x-y)}=\dfrac{y}{x-y}.$

10. $\dfrac{(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)}\times\dfrac{(c+a-b)(c-a+b)}{(c+a+b)(c-a-b)}=\dfrac{c-a+b}{c-a-b}.$

11. $\dfrac{(x-m+n)(x-m-n)}{(x-n+m)(x-n-m)}\times\dfrac{(x+n-m)(x-n+m)}{(x+m-n)(x-m+n)}=\dfrac{x-m+n}{x+m-n}.$

12. $\dfrac{(a+b+c+d)(a+b-c-d)}{(a+c+b+d)(a+c-b-d)}\times\dfrac{(a-b+d-c)(a-b-d+c)}{(a-c+d-b)(a-c-d+b)}=1.$

13. $\dfrac{(x^2-2xy+y^2)-z^2}{(x^2+2xy+y^2)-z^2}\times\dfrac{x+y-z}{x-y+z}=\dfrac{(x-y+z)(x-y-z)(x+y-z)}{(x+y+z)(x+y-z)(x-y+z)}=$ etc.

XLIII.

1. $\dfrac{2a}{x}\times\dfrac{5c}{3b}=\dfrac{10ac}{3bx}.$

2. $\dfrac{15y}{14z}\times\dfrac{7z}{5y^2}=\dfrac{3}{2y}.$

3. $\dfrac{8x^4y}{15ab^3}\times\dfrac{30ab^2}{2x^3}=\dfrac{8xy}{b}.$

4. $\dfrac{4a}{nx}\times\dfrac{1}{3ab}=\dfrac{4}{3bnx}.$

5. $\frac{3p}{2(p-1)} \times \frac{p-1}{2p} = \frac{3}{4}$.

6. $1 \times \frac{5x}{4a} = \frac{5x}{4a}$.

7. $\frac{5x}{7} \times \frac{1}{2} = \frac{5x}{14}$.

8. $\frac{1}{(x-1)(x-2)} \times \frac{x-1}{1} = \frac{1}{x-2}$.

9. $\frac{1}{(x-15)(x-2)} \times \frac{x-15}{1} = \frac{1}{x-2}$.

XLV.

1. $x^2 = x \times x$; $ax + x^2 = x(a+x)$; $\therefore$ L. C. M. is $x \times x(a+x)$.

2. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $\therefore$ L. C. M. is $(x+1)(x-1)x$.

3. $a^2 - b^2 = (a+b)(a-b)$; $a^2 + ab = a(a+b)$; $\therefore$ L. C. M. is $(a+b)(a-b)a$.

4. $4x^2 - 1 = (2x-1)(2x+1)$; $\therefore$ L. C. M. is $4x^2 - 1$.

5. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$; $\therefore$ L. C. M. is $a^3 + b^3$.

6. $x^2 - 1 = (x+1)(x-1)$; $\therefore$ L. C. M. is $x^2 - 1$.

7. $x^3 - 1 = (x-1)(x^2+x+1)$; $\therefore$ L. C. M. is $(x+1)(x^3-1)$.

8. $x^3 + 1 = (x+1)(x^2-x+1)$; $\therefore$ L. C. M. is $(x^2+1)(x^3+1)$.

9. $x^2 - 1 = (x+1)(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)(x^2+x+1)$.

10. $x^4 - 1 = (x^2+1)(x^2-1)$; $\therefore$ L. C. M. is $x^4 - 1$.

11. $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$; $x^3 + 1 = (x+1)(x^2-x+1)$;
$\therefore$ L. C. M. is $x(x-1)(x^2+x+1)(x+1)(x^2-x+1)$, etc.

12. $x^2 - 1 = (x+1)(x-1)$; $x^2 - x = x(x-1)$; $x^3 - 1 = (x-1)(x^2+x+1)$;
$\therefore$ L. C. M. is $(x+1)(x-1)x(x^2+x+1)$, etc.

13. $4a^2 - 1 = (2a+1)(2a-1)$; $8a^3 + 1 = (2a+1)(4a^2 - 2a + 1)$;
$\therefore$ L. C. M. is $(2a+1)(2a-1)(4a^2 - 2a + 1)$, etc.

14. $2x^2+2xy=2x(x+y)$; $\therefore$ L. C. M. is $2x^2+2xy$.

15. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is $(a+b)(a+b)(a-b)$.

16. $a^2-b^2=(a+b)(a-b)$; $\therefore$ L. C. M. is a^2-b^2.

17. $2(1-x^2)=2(1+x)(1-x)$; $\therefore$ L. C. M. is $4(1+x)(1-x)$.

18. $x^3-1=(x-1)(x^2+x+1)$; $\therefore$ L. C. M. is x^3-1.

19. L. C. M. is $(a-b)(a-c)(b-c)$.

20. L. C. M. is $(x+1)(x+2)(x+3)$.

21. $x^2-y^2=(x+y)(x-y)$;
$\therefore$ L. C. M. is $(x+y)(x-y)(x+y)(x-y)$.

22. $a^2-1=(a+1)(a-1)$; $\therefore$ L. C. M. is $(a+3)(a+1)(a-1)$.

23. $x(x^2-y^2)=x(x+y)(x-y)$; $\therefore$ L. C. M. is $x^2(x-y)(x+y)$.

24. L. C. M. is $(x+1)(x+3)(x+2)(x+4)$.

25. $x^2-y^2=(x+y)(x-y)$; $12(x^3+y^3)=12(x+y)(x^2-xy+y^2)$;
$\therefore$ L. C. M. is $12(x+y)(x-y)(x-y)(x^2-xy+y^2)$.

26. $6(x^2+xy)=6x(x+y)$; $8(xy-y^2)=8y(x-y)$;
$10(x^2-y^2)=10(x+y)(x-y)$;
$\therefore$ L. C. M. is $120xy(x+y)(x-y)$.

XLVI.

1. $x^2+5x+6=(x+2)(x+3)$
$x^2+6x+8=(x+2)(x+4)$.

2. $a^2-a-20=(a-5)(a+4)$
$a^2+a-12=(a-3)(a+4)$.

3. $x^2+3x+2=(x+1)(x+2)$
$x^2+4x+3=(x+1)(x+3)$.

4. $x^2+11x+30=(x+5)(x+6)$
$x^2+12x+35=(x+5)(x+7)$.

5. $x^2-\ 9x-22=(x-11)(x+2)$
$x^2-13x+22=(x-11)(x-2)$.

6. $2x^2+3x+1=(2x+1)(x+1)$
$x^2-x-2=(x-2)(x+1)$.

7. $x^3+x^2y+xy+y^2=(x^2+y)(x+y)$
$x^4-y^4=(x^2+y^2)(x+y)(x-y)$.

8. $x^2-8x+15=(x-3)(x-5)$
$x^2+2x-15=(x-3)(x+5)$.

9. $21x^2-26x+8)\,21x^3-12x^2-63x+36\,(x$

$$\begin{array}{l}
\underline{21x^3-26x^2+8x} \\
\quad 14x^2-71x+36 \\
\quad \underline{3} \\
\quad 42x^2-213x+108\,(2 \\
\quad \underline{42x^2-52x+16} \\
\qquad\quad -161x+92
\end{array}$$

Divide by -23; $7x-4)\,21x^2-26x+8\,(3x-2$

$\therefore$ L. C. M. is $(3x-2)(7x^3-4x^2-21x+12)$
or $(3x-2)(7x-4)(x^2-3)$.

10. $x^3+x^2y+xy^2+y^3=(x^2+y^2)(x+y)$
$x^3-x^2y+xy^2-y^3=(x^2+y^2)(x-y)$.

11. $a^3-2a^2b-ab^2+2b^3)\,a^3+2a^2b-ab^2-2b^3\,(1$

$$\begin{array}{l}
\underline{a^3-2a^2b-ab^2+2b^3} \\
\qquad 4a^2b \qquad -4b^3
\end{array}$$

Divide by $4b$; $a^2-b^2)\,a^3-2a^2b-ab^2+2b^3\,(a-2b$

$\therefore$ L. C. M. is $(a-2b)(a^3+2a^2b-ab^2-2b^3)$
or $(a-2b)(a^2-b^2)(a+2b)$.

XLVII.

1. $x^2-3x+2=(x-1)(x-2)$
$x^2-4x+3=(x-1)(x-3)$
$x^2-5x+4=(x-1)(x-4).$

2. $x^2+5x+4=(x+1)(x+4)$
$x^2+4x+3=(x+1)(x+3)$
$x^2+7x+12=(x+3)(x+4).$

3. $x^2-\ \ 9x+20=(x-4)(x-5)$
$x^2-12x+35=(x-7)(x-5)$
$x^2-11x+28=(x-4)(x-7).$

4. $$\begin{array}{l}6x^2-x-2)\,42x^2-34x+\ \ 4\,(7\\ \underline{42x^2-\ \ 7x-14}\\ -27x+18\end{array}$$

Divide by -9; $3x-2)\,6x^2-x-2\,(2x+1.$

Hence L. C. M. of first two expressions is

$(2x+1)(21x^2-17x+2)$, or $42x^3-13x^2-13x+2.$

We have now to find the L. C. M. of this and $14x^2+5x-1$.

$$\begin{array}{l}14x^2+5x-1)\,42x^3-13x^2-13x+2\,(3x-2\\ \underline{42x^3+15x^2-\ \ 3x}\\ -28x^2-10x+2\\ \underline{-28x^2-10x+2}\end{array}$$

$\therefore$ L. C. M. is $(3x-2)(14x^2+5x-1)$, etc.

5. $x^2-1=(x+1)(x-1)$
$x^2+2x-3=(x+3)(x-1)$
$6x^2-x-2=(3x-2)(2x+1).$

6. $x^3-27=(x-3)(x^2+3x+9)$
$x^2-15x+36=(x-3)(x-12)$
$x^3-3x^2-2x+6=x^2(x-3)-2(x-3)=(x^2-2)(x-3).$

XLVIII.

1. L. C. D. 20; new numerators $15x$, $16x$.
2. L. C. D. 18; new numerators $9x-21$, $4x-9$.
3. L. C. D. $10x^2$; new numerators $4x-8y$, $3x^2-8xy$.
4. L. C. D. $10a^2$; new numerators $20a+25b$, $6a^2-8ab$.
5. L. C. D. $60a^2c$; new numerators $48a^2-60ac$, $15a-10c$.
6. L. C. D. a^3b^2; new numerators $ab-b^2$, a^4-a^3b.
7. L. C. D. $1-x^2$; new numerators $3-3x$, $3+3x$.
8. L. C. D. $1-y^4$; new numerators $2+2y^2$, $2-2y^2$.
9. L. C. D. $1-x^2$; new numerators $5+5x$, 6.
10. L. C. D. $c(b+x)$; new numerators $ab+ax$, b.
11. L. C. D. $(a-b)(b-c)(a-c)$; new numerators $a-c$, $b-c$.
12. L. C. D. $abc(a-b)(a-c)(b-c)$; new numerators $bc-c^2$, $ab-b^2$.

XLIX.

1. $$\frac{12x+21}{15}+\frac{3x-4}{15}=\frac{15x+17}{15}.$$

2. $$\frac{36a-48b}{84}-\frac{56a-28b+28c}{84}+\frac{91a-28c}{84}=\frac{71a-20b-56c}{84}.$$

3. $$\frac{24x-18y}{42}+\frac{9x+21y}{42}-\frac{10x-4y}{42}+\frac{9x+2y}{42}=\frac{32x+9y}{42}.$$

4. $$\frac{30x-20y}{50x}+\frac{25x-35y}{50x}+\frac{16x^2+4xy}{50x}=\frac{16x^2+55x+4xy-55y}{50x}.$$

5. $$\frac{16x^2-28y^2}{12x^2}+\frac{6x^2-16xy}{12x^2}+\frac{5x^2-2x^2y}{12x^2}=\frac{27x^2-2x^2y-16xy-28y^2}{12x^2}.$$

6. $$\frac{180a^2+225b^2}{90b^2}+\frac{54ab+36b^2}{90b^2}+\frac{70b^2-20ab^2}{90b^2}=\frac{180a^2+54ab+331b^2-20ab^2}{90b^2}.$$

7. $$\frac{80x^3+100x^2}{60x^2}-\frac{36x^2-84x}{60x^2}+\frac{45}{60x^2}=\frac{80x^3+64x^2+84x+45}{60x^2}.$$

8. $$\frac{70a^2+28ab}{42ac}-\frac{84c^2-63bc}{42ac}+\frac{18ab-21bc}{42ac}=\frac{70a^2+46ab+42bc-84c^2}{42ac}, \text{ etc.}$$

9. $$\frac{2ac+5c^2}{a^2c^2}+\frac{4a^2c-3ac^2}{a^2c^2}-\frac{5ac-2c^2}{a^2c^2}=\frac{4a^2c-3ac^2-3ac+7c^2}{a^2c^2}, \text{ etc.}$$

10. $$\frac{3x^2y^2-4xy}{x^3y^3}-\frac{5x^2y^2+7x^2}{x^3y^3}-\frac{6x^2y^2-11y^2}{x^3y^3}=\frac{11y^2-8x^2y^2-4xy-7x^2}{x^3y^3}.$$

11. $$\frac{abc^2-b^2c^2}{a^3b^2c^2}+\frac{4a^2bc-5ab^2c}{a^3b^2c^2}+\frac{3a^4-7a^3b}{a^3b^2c^2}=\frac{3a^4-7a^3b+4a^2bc-5ab^2c+abc^2-b^2c^2}{a^3b^2c^2}.$$

L.

1. $$\frac{x+5+x-6}{(x-6)(x+5)}=\frac{2x-1}{(x-6)(x+5)}.$$

2. $$\frac{x-3-(x-7)}{(x-7)(x-3)}=\frac{4}{(x-7)(x-3)}.$$

3. $$\frac{1-x+1+x}{(1+x)(1-x)}=\frac{2}{(1+x)(1-x)}.$$

4. $$\frac{x^2+2xy+y^2-(x^2-2xy+y^2)}{(x+y)(x-y)}=\frac{4xy}{(x+y)(x-y)}.$$

5. $$\frac{1+x-2}{(1+x)(1-x)}=\frac{x-1}{(1+x)(1-x)}=\frac{-(1-x)}{(1+x)(1-x)}=\frac{-1}{1+x}$$

6. $$\frac{a(c+dx)-(ad-bc)x}{c(c+dx)}=\frac{ac+bcx}{c(c+dx)}=\frac{a+bx}{c+dx}.$$

7. $$\frac{x^2-xy+x^2+xy}{(x+y)(x-y)}=\frac{2x^2}{(x+y)(x-y)}.$$

8. $\dfrac{x-y+x}{(x-y)^2}=\dfrac{2x-y}{(x-y)^2}.$

9. $\dfrac{2(x+a)+3a}{(x+a)^2}=\dfrac{2x+5a}{(x+a)^2}$

10. $\dfrac{a-x+a+x}{2a(a+x)(a-x)}=\dfrac{2a}{2a(a+x)(a-x)}=\dfrac{1}{(a+x)(a-x)}.$

LI.

1. $\dfrac{1-a+1+a}{1-a^2}+\dfrac{2a}{1-a^2}=\dfrac{2+2a}{1-a^2}=\dfrac{2(1+a)}{1-a^2}=\dfrac{2}{1-a}.$

2. $\dfrac{1+x-(1-x)}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x}{1-x^2}+\dfrac{2x}{1+x^2}=\dfrac{2x+2x^3+2x-2x^3}{1-x^4}=\dfrac{4x}{1-x^4}.$

3. $\dfrac{x+x^2-x^2}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x}{1-x^2}+\dfrac{x}{1+x^2}=\dfrac{x+x^3+x-x^3}{1-x^4}=\dfrac{2x}{1-x^4}.$

4. $\dfrac{a+b-(a-b)}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2b}{a^2-b^2}-\dfrac{2b}{a^2+b^2}=\dfrac{2a^2b+2b^3-(2a^2b-2b^3)}{a^4-b^4}$

$$\frac{4b^3}{a^4-b^4}-\frac{4b^3}{a^4+b^4}=\frac{4a^4b^3+4b^7-(4a^4b^3-4b^7)}{a^8-b^8}=\frac{8b^7}{a^8-b^8}.$$

5. $\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x^2}{x(x+y)}=\dfrac{x^2+xy+y^2}{y(x+y)}+\dfrac{x}{x+y}=\dfrac{x^2+xy+y^2+xy}{y(x+y)}$

$$=\frac{(x+y)^2}{y(x+y)}=\frac{x+y}{y}.$$

6. $\dfrac{x^2-9+x^2-16}{(x+4)(x-3)}+\dfrac{x+5}{x+7}=\dfrac{2x^2-25}{(x+4)(x-3)}+\dfrac{x+5}{x+7}$

$$=\frac{2x^3-25x+14x^2-175+(x+5)(x^2+x-12)}{(x+4)(x-3)(x+7)}$$

$$=\frac{2x^3-25x+14x^2-175+x^3+6x^2-7x-60}{(x+4)(x-3)(x+7)},\text{ etc.}$$

7. $$\frac{x^2-4x+3+x^2-4x+4}{(x-2)(x-3)}+\frac{x-3}{x-4}=\frac{2x^2-8x+7}{(x-2)(x-3)}+\frac{x-3}{x-4}$$

$$=\frac{2x^3-16x^2+39x-28+(x-3)(x^2-5x+6)}{(x-2)(x-3)(x-4)}$$

$$=\frac{2x^3-16x^2+39x-28+x^3-8x^2+21x-18}{(x-2)(x-3)(x-4)}\text{, etc.}$$

8. $$\frac{3x-3a+4a}{(x-a)^2}-\frac{5a^2}{(x-a)^3}=\frac{(3x+a)(x-a)-5a^2}{(x-a)^3}=\frac{3x^2-2ax-a^2-5a^2}{(x-a)^3}$$

$$=\frac{3x^2-2ax-6a^2}{(x-a)^3}.$$

9. $$\frac{x+2-(x-1)}{(x-1)(x+2)}-\frac{3}{(x+1)(x+2)}=\frac{3}{(x-1)(x+2)}-\frac{3}{(x+1)(x+2)}$$

$$=\frac{3x+3-(3x-3)}{(x-1)(x+1)(x+2)}=\frac{6}{(x-1)(x+1)(x+2)}.$$

10. $$\frac{x+3}{(x+1)(x+2)(x+3)}-\frac{3}{(x+1)(x+2)(x+3)}=\frac{x}{(x+1)(x+2)(x+3)}.$$

11. $$\frac{x^2}{x^2-1}+\frac{x^2+x}{x^2-1}+\frac{x^2-x}{x^2-1}=\frac{3x^2}{x^2-1}.$$

12. $$\frac{a+e}{(a+c)(a+d)(a+e)}-\frac{a+d}{(a+c)(a+d)(a+e)}=\frac{e-d}{(a+c)(a+d)(a+e)}.$$

13. $$\frac{(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)}{(b+c)(c+a)(a+b)}$$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b+c)(c+a)(a+b)}=0.$$

14. $$\frac{(x-a)^2+(x-b)^2-(a-b)^2}{(x-a)(x-b)}=\frac{x^2-2ax+a^2+x^2-2bx+b^2-a^2+2ab-b^2}{(x-a)(x-b)}$$

$$=\frac{2(x^2-ax-bx+ab)}{x^2-ax-bx+ab}=2.$$

15. $$\frac{(x+y)^2-2xy}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}=\frac{x^2+y^2}{y(x+y)}+\frac{x^2y-x^3}{y(x^2-y^2)}$$

$$=\frac{(x^2+y^2)(x-y)+x^2y-x^3}{y(x^2-y^2)}=\frac{x^3+xy^2-x^2y-y^3+x^2y-x^3}{y(x^2-y^2)}$$

$$=\frac{y^2(x-y)}{y(x^2-y^2)}=\frac{y}{x+y}.$$

16. $$\frac{(a+b)(a-b)+(b+c)(b-c)+(c+a)(c-a)}{(b-c)(c-a)(a-b)}$$

$$=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(b-c)(c-a)(a-b)}=0.$$

17. $$\frac{x(x-y)}{x^3-y^3}+\frac{2xy}{x^3-y^3}=\frac{x^2+xy}{x^3-y^3}.$$

18. $$\frac{2b-2c+2a-2b}{(a-b)(b-c)}+\frac{2}{c-a}=\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)};$$

$$\frac{(2a-2c)(c-a)+2(a-b)(b-c)}{(a-b)(b-c)(c-a)}+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$$

$$=\frac{(2ac-2a^2-2c^2+2ac+2ab-2b^2-2ac+2bc)+(a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2)}{(a-b)(b-c)(c-a)}$$

$$=0.$$

19. $$\frac{(a+b)^2-2ab}{b(a+b)}+\frac{a^2b-a^3}{b(a^2-b^2)}=\frac{(a^2+b^2)(a-b)+a^2b-a^3}{b(a^2-b^2)}$$

$$=\frac{a^3+ab^2-a^2b-b^3+a^2b-a^3}{b(a^2-b^2)}=\frac{b^2(a-b)}{b(a^2-b^2)}=\frac{b}{a+b}.$$

20. $$\frac{(n+3)-1-(n+2)}{(n+1)(n+2)(n+3)}=0.$$

21. $$\frac{(a^2-bc)(b+c)+(b^2-ac)(a+c)+(c^2-ab)(a+b)}{(a+b)(a+c)(b+c)}$$

$$=\frac{a^2b+a^2c-b^2c-bc^2+ab^2-a^2c+b^2c-ac^2+ac^2-a^2b+bc^2-ab^2}{(a+b)(a+c)(b+c)}$$

$$=0.$$

LII.

1. $$\frac{x}{x-y}+\frac{x-y}{y-x}=\frac{x}{x-y}+\frac{-x+y}{x-y}=\frac{y}{x-y}.$$

2. $$\frac{(3+2x)(2+x)}{4-x^2}-\frac{(2-3x)(2-x)}{4-x^2}+\frac{x^2-16x}{4-x^2}$$
$$=\frac{6+7x+2x^2-(4-8x+3x^2)+x^2-16x}{4-x^2}=\frac{2-x}{4-x^2}=\frac{1}{2+x}.$$

3. $$\frac{x}{x+1}-\frac{-x}{x-1}+\frac{x^2}{x^2-1}=\frac{x^2-x-(-x^2-x)+x^2}{x^2-1}=\frac{3x^2}{x^2-1}.$$

4. $$\frac{1}{6(y+1)}-\frac{1}{2(y-1)}+\frac{4}{3(1-y^2)}=\frac{y-1-3(y+1)}{6(y^2-1)}+\frac{-4}{3(y^2-1)}$$
$$=\frac{y-1-3y-3-8}{6(y^2-1)}=\frac{-2y-12}{6(y^2-1)}=\frac{2y+12}{6(1-y^2)}=\frac{y+6}{3(1-y^2)}.$$

5. $$\frac{1}{(m-2)(m-3)}+\frac{-2}{(m-1)(m-3)}+\frac{1}{(m-1)(m-2)}$$
$$=\frac{m-1-2(m-2)+m-3}{(m-1)(m-2)(m-3)}=0.$$

6. $$\frac{1}{(a-b)(x+b)}+\frac{-1}{(a-b)(x+a)}=\frac{x+a-(x+b)}{(a-b)(x+a)(x+b)}$$
$$=\frac{a-b}{(a-b)(x+a)(x+b)}=\frac{1}{(x+a)(x+b)}.$$

7. $$\frac{a^2+b^2}{(a+b)(a-b)}-\frac{2ab^2}{(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{(a^2+b^2)(a^2+ab+b^2)-2ab^2(a+b)}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{a^4+a^2b^2+a^3b+ab^3+a^2b^2+b^4-2a^2b^2-2ab^3}{(a+b)(a-b)(a^2+ab+b^2)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$
$$=\frac{a^4+a^3b-ab^3+b^4}{(a+b)(a^3-b^3)}+\frac{2a^2b}{(a+b)(a^2-ab+b^2)}$$

$$=\frac{(a^4+a^3b-ab^3+b^4)(a^2-ab+b^2)+2a^2b(a^3-b^3)}{(a+b)(a^3-b^3)(a^2-ab+b^2)}$$

$$=\frac{a^6+2a^2b^4-2ab^5+b^6+2a^5b-2a^2b^4}{(a^3-b^3)(a^3+b^3)}=\frac{a^6+2a^5b-2ab^5+b^6}{a^6-b^6}.$$

8. $$\frac{1}{4(1+x)}-\frac{-1}{4(1-x)}+\frac{1}{2(1+x^2)}=\frac{1-x-(-1-x)}{4(1-x^2)}+\frac{1}{2(1+x^2)}$$

$$=\frac{2}{4(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1}{2(1-x^2)}+\frac{1}{2(1+x^2)}=\frac{1+x^2+1-x^2}{2(1-x^4)}$$

$$=\frac{2}{2(1-x^4)}=\frac{1}{1-x^4}.$$

9. $$\frac{1}{(x-y)(y-z)}+\frac{-1}{(x-y)(x-z)}+\frac{1}{(x-z)(y-z)}$$

$$=\frac{(x-z)+(-y+z)+(x-y)}{(x-y)(y-z)(x-z)}=\frac{2(x-y)}{(x-y)(y-z)(x-z)}=\frac{2}{(x-z)(y-z)}.$$

10. $$\frac{1}{a(a-b)(a-c)}+\frac{-1}{b(a-b)(b-c)}+\frac{1}{c(a-c)(b-c)}$$

$$=\frac{bc(b-c)+ac(-a+c)+ab(a-b)}{abc(a-b)(a-c)(b-c)}$$

$$=\frac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}=\frac{1}{abc}.$$

LIII.

1. $$\frac{1}{(x+4)(x+5)}+\frac{1}{(x+7)(x+5)}=\frac{x+7+x+4}{(x+4)(x+5)(x+7)}$$

$$=\frac{2x+11}{(x+4)(x+5)(x+7)}.$$

2. $$\frac{1}{(x-7)(x-6)}+\frac{1}{(x-9)(x-6)}=\frac{x-9+x-7}{(x-7)(x-6)(x-9)}$$

$$=\frac{2(x-8)}{(x-6)(x-7)(x-9)}.$$

3. $$\frac{1}{(x+11)(x-4)}+\frac{1}{(x+11)(x-13)}=\frac{x-13+x-4}{(x-4)(x+11)(x-13)}$$
$$=\frac{2x-17}{(x-4)(x+11)(x-13)}.$$

4. $$\frac{1}{(x+1)(x+2)}+\frac{2x}{(x+1)(x+3)}+\frac{1}{(x+2)(x+3)}$$
$$=\frac{x+3+2x(x+2)+x+1}{(x+1)(x+2)(x+3)}=\frac{2x^2+6x+4}{(x+1)(x+2)(x+3)}$$
$$=\frac{2(x^2+3x+2)}{(x^2+3x+2)(x+3)}=\frac{2}{x+3}.$$

5. $$\frac{m^2+mn+2mn}{n(m+n)}-\frac{2mn}{(m+n)^2}=\frac{(m^2+3mn)(m+n)-2mn^2}{n(m+n)^2}$$
$$=\frac{m^3+3m^2n+m^2n+3mn^2-2mn^2}{n(m+n)^2}=\frac{m^3+4m^2n+mn^2}{n(m+n)^2}.$$

6. $$\frac{(1+x)(1-x+x^2)+(1-x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)}-\frac{2}{1+x^2+x^4}$$
$$=\frac{1+x^3+1-x^3}{1+x^2+x^4}-\frac{2}{1+x^2+x^4}=0.$$

7. $$\frac{5+5x-6+6x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}-\frac{-7x}{3(1-x^2)}=\frac{11x-1+7x}{3(1-x^2)}+\frac{7x}{3(1+x^2)}$$
$$=\frac{(18x-1)(1+x^2)+7x(1-x^2)}{3(1-x^4)}=\frac{18x+18x^3-1-x^2+7x-7x^3}{3(1-x^4)}$$
$$=\frac{11x^3-x^2+25x-1}{3(1-x^4)}.$$

8. $$\frac{(3-x)+2(x-1)}{8(x-1)(3-x)}+\frac{1}{8(x-5)}+\frac{1}{(1-x)(x-3)(x-5)}$$
$$=\frac{(x+1)(x-5)+(x-1)(3-x)}{8(x-1)(3-x)(x-5)}+\frac{1}{(x-1)(3-x)(x-5)}$$

$$=\frac{x^2-4x-5+4x-x^2-3+8}{8(x-1)(3-x)(x-5)}=0.$$

9. $\dfrac{(1-x+x^2-x^3)(1+x)+x^4}{1+x}=\dfrac{1-x^4+x^4}{1+x}=\dfrac{1}{1+x}.$

LIV.

1. Multiply by 2 ; $x=16$.

2. Multiply by 4 ; $3x=36$, $x=12$.

3. Multiply by 15 ; $5x+3x=120$; $8x=120$, $x=15$.

4. Multiply by 28 ; $7x-4x=84$; $3x=84$, $x=28$.

5. $-\dfrac{4x}{9}=-28$; $\dfrac{4x}{9}=28$, $\dfrac{x}{9}=7$, $x=63$.

6. Multiply by 15 ; $10x=528-12x$; $22x=528$; $x=24$.

7. Multiply by 12 ; $8x+48=7x+108$; $x=60$.

8. $\dfrac{2x}{3}+6=\dfrac{4x}{5}$; $10x+90=12x$; $x=45$.

9. $\dfrac{3x}{4}+3=\dfrac{5x}{6}$; $9x+36=10x$; $x=36$.

10. $\dfrac{7x}{8}=\dfrac{9x}{10}-3$; $35x=36x-120$; $x=120$.

11. $\dfrac{5x}{9}+\dfrac{7x}{12}=82$; $20x+21x=82\times 36$; $x=72$.

12. $\dfrac{x}{6}+\dfrac{x}{8}=28$; $4x+3x=28\times 24$; $x=96$.

13. $\dfrac{5x}{8}-\dfrac{3x}{4}=-8$; $5x-6x=-64$; $x=64$.

14. $9x+360-10x=348$; $-x=-12$; $x=12$.

15. $3x-44=2x-16$; $x=28$.

16. $6x+4x+3x=13$; $13x=13$; $x=1$.

17. Multiply by 70; $14x+28+10x-10=35x-70$; $-11x=-88$; $x=8$.

18. Multiply by 12; $6x+4x=117-3x$; $13x=117$; $x=9$.

19. Multiply by 140; $35x+315+40x=84x-168+420$; $-9x=-63$; $x=7$.

20. Multiply by 105; $357-63x=1015-385x+140x+70$; $182x=728$; $x=4$.

21. Multiply by 7; $2x-10=0$; $2x=10$; $x=5$.

22. Multiply by 329; $141x+188+28x-357=0$; $169x=169$; $x=1$.

23. Multiply by x; $3-3x=1-x$; $-2x=-2$; $x=1$.

24. Multiply by x; $12+x-5x=6$; $-4x=-6$; $x=\frac{3}{2}$.

25. Multiply by 20; $5x+2x+x=800$; $8x=800$; $x=100$.

26. Multiply by 8; $18x+12-4x=29x-348$; $-15x=-360$; $x=24$.

27. Multiply by $100x$; $275x-300=100-325x$; $600x=400$; $x=\frac{2}{3}$.

28. Multiply by 90; $225+540-30x=100x+30+27-18x+36$; $-112x=-672$; $x=6$.

29. Multiply by 12; $4x+3x-10x-144=20x-696$; $-23x=-552$; $x=24$.

30. Multiply by 60; $42x+12-720-45x=36x+156-255x$; $216x=864$; $x=4$.

LV.

1. Multiply by 2; $10x - x - 2 = 142$; $9x = 144$; $x = 16$.
2. Multiply by 3; $3x - 3 + x = 17$; $4x = 20$; $x = 5$.
3. Multiply by 4; $5 - 2x + 8 = 4x - 12x + 16$; $6x = 3$; $x = \frac{1}{2}$.
4. Multiply by 4; $10x - 5x = 9 - 6 + 2x$; $3x = 3$; $x = 1$.
5. Multiply by 30; $60x - 25x + 20 = 210 - 6 + 12x$; $23x = 184$; $x = 8$.
6. Multiply by 36; $18x + 36 = 56 - 27 - 45x$; $63x = -7$; $x = -\frac{1}{9}$.
7. Multiply by 48; $30x + 18 - 48 + 64x + 24x = 744 - 72 + 40x$; $78x = 702$; $x = 9$.
8. Multiply by 385; $55x + 275 - 77x + 154 = 35x + 315$; $-57x = -114$; $x = 2$.
9. Multiply by 105; $35x + 35 - 15x + 60 = 21x + 84$; $-x = -11$; $x = 11$.
10. Multiply by 24; $24x - 72 - 3x - 6 = 8x$; $13x = 78$; $x = 6$.
11. Multiply by 84; $12x + 60 = 21x + 42 - 28x + 56$; $19x = 38$; $x = 2$.
12. Multiply by 33; $11x - 3x + 3 = 33x - 297$; $-25x = -300$; $x = 12$.
13. Multiply by 70; $14x + 28 = 35x - 70 - 10x + 10$; $-11x = -88$; $x = 8$.
14. Multiply by 140; $35x + 315 - 84x + 168 = 420 - 40x$; $-9x = -63$; $x = 7$.
15. Multiply by 78; $39x + 39 - 26x + 78 = 6x + 180$; $7x = 63$; $x = 9$.
16. Multiply by 35; $10x - 7x - 21 = 105x - 735$; $-102x = -714$; $x = 7$.

17. Multiply by 154 ; $44x + 154 - 126x + 112 = 77x - 847$; $-159x = -1113$; $x = 7$.

18. Multiply by 572 ; $1001x - 4433 - 176 - 330x = 182x - 208$; $489x = 4401$; $x = 9$.

19. Multiply by 273 ; $728x - 1365 - 429x + 39 = 147x + 42$; $152x = 1368$; $x = 9$.

20. Multiply by 56 ; $49x + 63 - 24x - 8 = 126x - 182 - 996 + 36x$; $-137x = -1233$; $x = 9$.

21. Multiply by 280 ; $28x + 2800x = 140x + 56x + 7x - 400 + 40x + 26250$; $2585x = 25850$; $x = 10$.

LVI.

1. $(a + b)x = c$; $x = \dfrac{c}{a + b}$.

2. $5bx - cx = 3c - 2a$; $(5b - c)\,x = 3c - 2a$, etc.

3. $ax + fx = a^2b - bc + d$; $(a + f)\,x = a^2b - bc + d$, etc.

4. $ax - 5x = bc - dm$; $(a - 5)x = bc - dm$, etc.

5. $-a^2x - ax = -a^2b - abc$; $ax + x = ab + bc$; $(a + 1)\,x = b\,(a + c)$, etc.

6. $3acx - 12cdx = abc + 6bcd$; $(3a - 12d)\,x = ab + 6bd$, etc.

7. $3ackx - kx + ackx = -k^2 - 3k - k^2 + 3abk$;
   $3acx - x + acx = -k - 3 - k + 3ab$; $(4ac - 1)x = 3ab - 2k - 3$;
   $x = \dfrac{3ab - 2k - 3}{4ac - 1}$.

8. $abcx - cmx + ac^2x = ac^2 + abc - mc$;
   $abx - mx + acx = ac + ab - m$; $x = 1$.

9. $(a+b)^2-x^2=ab+(b-a)\,x-x^2-ab$; $x=\dfrac{(a+b)^2}{b-a}$.

10. $a^2-x^2=2a^2+2ax-x^2$; $-2ax=a^2$; $x=-\dfrac{a}{2}$.

11. $a^4+2a^2x+x^2=x^2+4a^2+a^4$; $2a^2x=4a^2$; $x=2$.

12. $a^4-x^2=a^4+2ax-x^2$; $-2ax=0$; $x=0$.

13. $ax-b+ac=x+ac$; $ax-x=b$; $x=\dfrac{b}{a-1}$.

14. $2ax-3a+bx=1$; $2ax+bx=3a+1$; $x=\dfrac{3a+1}{2a+b}$.

15. $18a-4ax+2b=3x$; $3x+4ax=18a+2b$; $x=\dfrac{18a+2b}{4a+3}$.

16. $ax^2-bx-1=ax^2-a$; $-bx=1-a$; $x=\dfrac{a-1}{b}$.

17. $mp^2x+mx^3=mpqx^2+mx^3$; $mp^2x=mpqx^2$; $p=qx$; $x=\dfrac{p}{q}$.

18. $dx-abd=ac-adx$; $dx+adx=abd+ac$; $x=\dfrac{abd+ac}{ad+d}$.

19. $x^2-a-ax+x^2=2x^2-ab$; $-ax=a-ab$; $x=b-1$

20. $3bx-abc+cx^2=4bx-abc$; $cx^2=bx$; $x=\dfrac{b}{c}$.

21. $a^3b+a^2x-b^3+bx=b^2x-b^3-a^3b+a^2x$; $bx-b^2x=-2a^3b$;

$$x=\frac{2a^3}{b-1}.$$

22. $6ax-4b-3ax+3a=6ax-4b$; $-3ax=-3a$; $x=1$.

23. $abm^2-b^2m-amx+bx=0$; $(b-am)x=bm\,(b-am)$; $x=bm$.

24. $2a^3b^4-b^3x+3a^3bc=3a^2cx\,(a+b)-ab^4+2a^2b^3x$;

$$-b^3x-3a^3cx-3a^2bcx-2a^2b^3x=-3a^3bc-2a^3b^4-ab^4;$$

$$x=\frac{3a^3bc+2a^3b^4+ab^4}{b^3+3a^3c+3a^2bc+2a^2b^3}.$$

25. $acx^2+abc-ac^2x+abx-acx^2=0$; $ac^2x-abx=abc$; $x=\dfrac{bc}{c^2-b}$.

26. $ad^2+ax^2=acdx+ax^2$; $x=\dfrac{d}{c}$.

27. $ab=(bc+d)\,x+1$; $x=\dfrac{ab-1}{bc+d}$.

28. $3ac+cx=3a^2+ax+am-mx$; $cx-ax+mx=am-3ac+3a^2$;

$$x=\frac{am-3ac+3a^2}{c-a+m}.$$

29. $ab+ax+bx+x^2-ab-ac=\dfrac{a^2c}{b}+x^2$; $ax+bx=ac+\dfrac{a^2c}{b}$;

$$(a+b)x=\frac{(ab+a^2)\,c}{b};\quad x=\frac{ac}{b}.$$

30. $a^2ce-a^2dx-2abdx-b^2dx-abdx=a^2de-3abdx$;

$$-a^2dx-b^2dx=a^2de-a^2ce;\quad x=\frac{a^2e\,(c-d)}{(a^2+b^2)\,d}.$$

LVII.

1. $(3x+7)(4x+3)=(3x+5)(4x+5)$;
$12x^2+37x+21=12x^2+35x+25$; $2x=4$; $x=2$.

2. $(x+6)(2x-5)=x(2x+5)$; $2x^2+7x-30=2x^2+5x$; $2x=30$; $x=15$.

3. $(2x+7)(2x-1)=(4x-1)(x+2)$; $4x^2+12x-7=4x^2+7x-2$;
$5x=5$; $x=1$.

4. $(5x-1)(2x-3)=(5x-3)(2x+3)$; $10x^2-17x+3=10x^2+9x-9$;

$$-26x=-12;\quad x=\frac{6}{13}.$$

5. $4x-3+2(3x-2)=0$; $4x-3+6x-4=0$; $10x=7$; $x=\dfrac{7}{10}$.

6. $2(1-2x)-5(1-5x)=0$; $2-4x-5+25x=0$; $21x=3$; $x=\frac{1}{7}$.

7. $\frac{x+1+x-1}{x^2-1}=\frac{3}{x^2-1}$; $2x=3$; $x=\frac{3}{2}$.

8. $\frac{7x-29}{5x-12}=\frac{8x+19}{18}-\frac{4x+3}{9}$; $\frac{7x-29}{5x-12}=\frac{8x+19-8x-6}{18}$;
$18(7x-29)=13(5x-12)$; $126x-522=65x-156$; $61x=366$;
$x=6$.

9. $\frac{x}{3}-\frac{2}{3}=\frac{x^2-5x}{3x-7}$; $(x-2)(3x-7)=3(x^2-5x)$;
$3x^2-13x+14=3x^2-15x$; $2x=-14$; $x=-7$.

10. $\frac{(3x+2)(x+2)+(2x-4)(x-1)}{x^2+x-2}=5$;
$3x^2+8x+4+2x^2-6x+4=5x^2+5x-10$; $-3x=-18$; $x=6$.

11. Multiply by 210; $35(x+3)-30(11-x)=84(x-4)-10(x-3)$;
$35x+105-330+30x=84x-336-10x+30$; $-9x=-81$;
$x=9$.

12. $(x+1)(2x+2)=2(x-3)(x+6)$; $2x^2+4x+2=2x^2+6x-36$;
$-2x=-38$; $x=19$.

13. $\frac{3x^2(2x+3)+2x+1}{6x^2+3x}=x+1$; $6x^3+9x^2+2x+1=(6x^2+3x)(x+1)$;
$6x^3+9x^2+2x+1=6x^3+9x^2+3x$; $-x=-1$; $x=1$.

14. $\frac{3(x-1)-(x+1)(x+1)}{x^2-1}=\frac{-x^2}{x^2-1}$; $3x-3-x^2-2x-1=-x^2$; $x=4$.

15. $\frac{2(1+x)+8(1-x)}{1-x^2}=\frac{45}{1-x^2}$; $2+2x+8-8x=45$; $-6x=35$;
$x=-\frac{35}{6}$.

16. $\frac{4}{x-8}+\frac{3}{2(x-8)}-\frac{29}{24}=\frac{2}{3(x-8)}$; multiply by $24(x-8)$;

$96+36-29x+232=16$; $-29x=-348$; $x=12$.

17. $x^4-4x^2+20x-24=x^4-4x^2+16x-16$; $4x=8$; $x=2$.

18. $2x^4+2x^3-9x^2+12=2x^4+2x^3-23x^2+7x+12$; $14x^2=7x$; $x=\frac{1}{2}$.

19. Multiply by $16x$; $4x^2-16x=4x^2-3x-\frac{13}{8}$; $-13x=-\frac{13}{8}$; $x=\frac{1}{8}$

20. $5-x\left(\frac{7x-4}{2x}\right)=\frac{x}{2}-\frac{8x-4}{4}$; $5-\frac{7x-4}{2}=\frac{x}{2}-\frac{4x-2}{2}$;

$10-7x+4=x-4x+2$; $-4x=-12$; $x=3$.

LVIII.

1. $\frac{5x}{10}-2=\frac{25x}{100}+\frac{2x}{10}-1$; $50x-200=25x+20x-100$; $x=20$.

2. $\frac{325x}{100}-\frac{51}{10}+x-\frac{75x}{100}=\frac{39}{10}+\frac{5x}{10}$; $325x-510+100x-75x=390+50x$

$300x=900$; $x=3$.

3. $\frac{125x}{1000}+\frac{x}{100}=13-\frac{2x}{10}+\frac{4}{10}$; $125x+10x=13000-200x+400$;

$335x=13400$; $x=40$.

4. $\frac{3x}{10}+\frac{1305x}{1000}+\frac{5x}{10}=\frac{2295}{100}-\frac{195x}{1000}$;

$300x+1305x+500x=22950-195x$; $2300x=22950$; $x=\frac{459}{46}$.

5. $\frac{2x}{10}-\frac{x}{100}+\frac{5x}{1000}=\frac{117}{10}$; $200x-10x+5x=11700$;

$195x=11700$; $x=60$.

6. $\frac{24x}{10}-\frac{36x-5}{50}=\frac{8x}{10}+\frac{89}{10}$; $120x-36x+5=40x+445$; $44x=440$;

$x=10$.

7. $\frac{24x}{10}-\frac{1075}{100}=\frac{25x}{100}$; $240x-1075=25x$; $215x=1075$; $x=5$.

8. $\frac{5x}{10}+2-\frac{75x}{100}=\frac{4x}{10}-11$; $50x+200-75x=40x-1100$;

$-65x=-1300$; $x=20$.

9. $\frac{405}{900x}=\cdot150$; $\frac{45}{100x}=\cdot15$; $45=15x$; $x=3$.

10. $\frac{25x}{10}-\frac{2+x}{7}\times\frac{-7}{4}=\frac{5}{10}-\frac{5x+3}{8}$; $\frac{5x}{2}+\frac{2+x}{4}=\frac{1}{2}-\frac{5x+3}{8}$;

$20x+4+2x=4-5x-3$; $27x=-3$; $x=-\frac{1}{9}$

11. $\frac{85}{20}-\frac{2}{10x}=\frac{17}{4}-\frac{10-x}{10x}$; $85x-4=85x-20+2x$; $-2x=-16$; $x=8$.

12. $\frac{48x}{600}-\frac{30-40x}{2}=1993$; $8x-1500+2000x=199300$;

$2008x=200800$; $x=100$.

13. $\frac{20-30x}{15}+\frac{500x}{125}-\frac{2x-3}{9}=\frac{10x-20}{18}+\frac{25}{9}$;

$\frac{4-6x}{3}+4x-\frac{2x-3}{9}=\frac{5x-10}{9}+\frac{25}{9}$;

$12-18x+36x-2x+3=5x-10+25$; $11x=0$; $x=0$.

14. $\frac{2408}{100x}+\frac{1}{x}\times\frac{4}{100}\times\frac{10x+9}{10}=\frac{2412}{10}$; $\frac{2408}{100x}+\frac{40x+36}{1000x}=\frac{2412}{10}$;

$24080+40x+36=241200x$; $24116=241160x$; $x=\cdot1$.

15. $\frac{5x}{10}+\frac{45x-75}{60}=\frac{12}{2}-\frac{3x-6}{9}$; $\frac{x}{2}+\frac{3x-5}{4}=6-\frac{x-2}{3}$;

$6x+9x-15=72-4x+8$; $19x=95$: $x=5$.

16. $\frac{1}{2}-\frac{35x}{10x-20}-\frac{24-3x}{8}=\frac{375x}{1000}$; $500-\frac{3500x}{x-2}-3000+375x=375x$;

$$-2500=\frac{3500x}{x-2}\text{ ; }-25x+50=35x\text{ ; }50=60x\text{ ; }x=\frac{5}{6}.$$

17. $\frac{15x}{100}+\frac{135x-225}{600}=\frac{36}{20}-\frac{9x-18}{90}$; $\frac{3x}{20}+\frac{45x-75}{200}=\frac{18}{10}-\frac{x-2}{10}$;

$$30x+45x-75=360-20x+40\text{ ; }95x=475\text{ ; }x=5.$$

LIX.

1. Let x be the number; then $\frac{x}{2}+\frac{x}{4}+\frac{x}{5}=95$, etc.

2. Let x be the number; then $\frac{x}{12}+\frac{x}{20}+\frac{x}{40}=38$, etc.

3. Let x be the number; then $\frac{x}{4}-\frac{x}{5}=4$, etc.

4. Let x be the number; then $\frac{x}{25}-\frac{x}{35}=8$, etc.

5. Let x be one part; then $60-x$ is the other; and $\frac{x}{7}=\frac{60-x}{8}$, etc.

6. Let x be one part; then $50-x$ is the other;

   and $\frac{x}{4}+\frac{5(50-x)}{6}=40$, etc.

7. Let x be the greater; then $100-x$ is the less; and

   $\frac{x}{4}-\frac{100-x}{3}=11$, etc.

8. Let x be the number; then $x-\left(\frac{x}{3}+\frac{x}{10}+\frac{x}{12}\right)=58$, etc.

9. Let x be the number; then $33-\frac{x}{4}-\frac{x}{5}-\frac{x}{10}=0$, etc.

10. Let x be the number; then $\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=\frac{x}{2}+112$, etc.

11. Let x be the number; then $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{12}+30=2x$, etc.

12. Let x be the greater; then $x-8$ is the less; and $\frac{x}{x-8}=3$, etc.

13. Let x be the property in pounds; then $\frac{x}{7}=x-1626$, etc.

14. Let x be the greater; then $x-504$ is the less; and $\frac{x}{x-504}=15$, etc.

15. Let x be the greater; then $5760-x$ is the less; and

$$x-(5760-x)=\frac{x}{3} \text{ etc.}$$

16. Let x be the number; then $x+\frac{x}{2}-60=65-x$, etc.

17. Let x be the greater; then $x-20$ is the less; and $\frac{x}{7}=\frac{x-20}{3}$, etc.

18. Let x be the less; then $31207-x$ is the greater; and

$$\frac{31207-x}{x}=15+\frac{1335}{x}, \text{ etc.}$$

19. Let x be the age of the younger; then $27-x$ is the age of the elder;

and $\frac{27-x}{x}=\frac{7}{2}$, etc.

20. Let x be the greater; then $237-x$ is the less; and $237-x=\frac{4x}{5}$, etc.

21. Let x be A's share; then $\frac{2x}{7}$ is B's share; and $x+\frac{2x}{7}=1800$, etc.

22. Let x be one part; then $46-x$ is the other; and $\frac{x}{7}+\frac{46-x}{3}=10$, etc.

23. Let x be one part; then $a-x$ is the other; and $\frac{x}{m}+\frac{a-x}{n}=b$, etc.

24. Let x be one number; then $a-x$ is the other; and $x-(a-x)=b$, etc.

25. Let x be the number; then $\frac{4x}{3}=24$, etc.

26. Let x be B's share; then $\frac{5x}{11}$ is A's share; and $x+\frac{5x}{11}$ is C's share; then $x+\frac{5x}{11}+x+\frac{5x}{11}=864$, etc.

27. Let x be the property; then $\frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600=x$, etc.

28. Let x be the second; then $70-x$ is the first; and $\frac{70-x}{x}=2+\frac{1}{x}$, etc.

29. Let x be the first; then $x+25$ is the second; and $\frac{x+25}{x}=4+\frac{4}{x}$.

30. Let x be the greater; then $208-x$ is the less; and $\frac{x}{4}+\frac{208-x}{3}=4\{x-(208-x)\}-4$, etc.

31. Let x be the number of days; then $\frac{2x}{3}-\frac{x}{2}=13$, etc.

32. Let x be the number of gallons; then $\frac{4x}{5}-10=\frac{2x}{3}$.

33. In 25 years the sum of their ages will be increased by 50 years; hence the sum of their ages at the present time is 50 years. Let x be the age of the father; then $50-x$ is the age of the

son; and the difference of their ages is $x-(50-x)$, or, $2x-50$. In 20 years the age of the father will be $x+20$, and the age of the son $70-x$. Hence the sum in 20 years will be $x+20+(70-x)$, or, 90 years. Therefore, by the question, $2x-50=30$, or, $x=40$.

34. Let x be the number of years; then $70-x=\frac{10}{3}(35-x)$, etc.

35. Let x be the number of years; then $72-x=5(48-x)$, etc.

36. Let x be the number of days; then $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$, etc.

37. Let x be the number of days; then $\frac{1}{50}+\frac{1}{60}+\frac{1}{75}=\frac{1}{x}$, etc.

38. Let x be the number of days; then $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ represents the part done daily by *two* men of A's strength, *two* of B's, and *two* of C's, working together. Hence A, B, and C together do $\frac{1}{2}$ of $\left(\frac{1}{12}+\frac{1}{15}+\frac{1}{20}\right)$ daily.

$$\therefore\ \frac{1}{x}=\frac{1}{2}\text{ of }\frac{5+4+3}{60},\text{ or, }x=\frac{120}{12}=10.$$

39. Let x be the number of hours; then $\frac{1}{x}$ is the part A does hourly;

A and B do $\frac{1}{4}$ hourly; A and C do $\frac{5}{18}$ hourly;

$\therefore$ B does $\frac{1}{4}-\frac{1}{x}$, and C does $\frac{5}{18}-\frac{1}{x}$ hourly.

$\therefore$ B and C do $\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$ hourly.

Hence $\frac{7}{36}=\frac{1}{4}+\frac{5}{18}-\frac{2}{x}$, etc.

40. Let x be the number of days; then $\frac{2}{5}+\frac{3}{10}+\frac{4}{15}=\frac{1}{x}$, etc.

41. Let x be the number of days; then B will do the whole in $3x$ days.

$\therefore$ B does $\frac{1}{3x}$ daily. Now A does $\frac{3}{50}$ daily, and in 3 days he does $\frac{9}{50}$, while B in 3 days does $\frac{1}{x}$.

Hence $\frac{9}{50}+\frac{1}{x}=\frac{2}{5}$, etc.

42. Let x be the number of hours; then $\frac{1}{3}+\frac{1}{4}=\frac{1}{x}$, etc.

43. Let x be the number of hours; then $\frac{3}{4}+\frac{3}{10}+\frac{1}{5}=\frac{1}{x}$, etc.

44. Let x be the number of minutes; then $\frac{1}{40}-\frac{1}{60}=\frac{1}{x}$, etc.

45. Let x be the number of minutes; then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$, etc.

46. The first pipe lets in 12 gall. in $\frac{13}{4}$ min., or, $\frac{48}{13}$ gall. in a min.

The second „ $\frac{46}{3}$ gall. in $\frac{5}{2}$ min., or, $\frac{92}{15}$ gall. in a min.

The third „ 17 gall. in 3 min., or, $\frac{17}{3}$ gall. in a min.

$\therefore$ All three together let in $\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)$ gall. in a min.

Let x be the number of minutes in which they fill the vessel;

then $x\left(\frac{48}{13}+\frac{92}{15}+\frac{17}{3}\right)=755\frac{1}{4}$,

or, $\frac{2160x+3588x+3315x}{585}=\frac{3021}{4}$, or, $\frac{9063x}{585}=\frac{3021}{4}$,

or $\frac{3x}{585}=\frac{1}{4}$; $\therefore x=\frac{585}{12}=48\frac{3}{4}$.

47. Let x be the number of gallons let in per minute by the third pipe; then $x+10$ and $x-4$ are the number of gallons let in by the other pipes; then $15(x+x+10+x-4)=2400$, etc.

48. Let x be the distance from Ely; then $16-x$ is the distance from Cambridge; and $\frac{x}{4\frac{4}{9}}=\frac{16-x}{\frac{60}{18}}$; $\frac{9x}{40}=\frac{9(16-x)}{30}$; $3x=64-4x$; $x=9\frac{1}{7}$.

49. Let x be the distance up; then x is the distance down; and $\frac{x}{2\frac{1}{3}}+\frac{x}{3\frac{1}{2}}=5$; $\frac{3x}{7}+\frac{2x}{7}=5$; $x=7$; and $\therefore 2x=14$.

50. In 1 hour he walks $\frac{a}{b}$ miles; hence in c hours he walks $\frac{ac}{b}$ miles. He walks 1 mile in $\frac{b}{a}$ hours; hence he walks d miles in $\frac{bd}{a}$ hours.

51. Let x be the number of days; then $286x=244(x+2)$, etc.

52. Let x be the number of hours; then $x\times\frac{22\frac{1}{2}}{3}=(x+8)\times\frac{31\frac{1}{2}}{5}$; $\frac{15x}{2}=\frac{63(x+8)}{10}$, etc.

53. Let x be the distance from Cambridge; $60-x$ is the distance from London; then $\frac{x}{4}=\frac{60-x}{3\frac{3}{4}}$; $\frac{x}{4}=\frac{240-4x}{15}$, etc.

54. Let x be the number of hours; then $\frac{5}{3}\times x=\frac{7}{5}(x+8)$; $x=42$. Hence 50 hours after A started.

55. (1.) Let the time be x minutes past 1; then $x=5+\frac{x}{12}+30$, etc.

(2.) Let the time be x minutes past 4; then $x=20+\frac{x}{12}+30$, etc.

(3.) Let the time be x minutes past 8; then $x+30=40+\frac{x}{12}$, etc.

56. (1.) Let the time be x minutes past 2; then $x = 10 + \frac{x}{12} + 15$, etc.

(2.) Let the time be x minutes past 4; then there are two solutions, (α) $x + 15 = 20 + \frac{x}{12}$, etc. ($\beta$) $x = 20 + \frac{x}{12} + 15$, etc.

(3.) Let the time be x minutes past 7; then there are two solutions, (α) $x + 15 = 35 + \frac{x}{12}$, etc. ($\beta$) $x = 35 + \frac{x}{12} + 15$, etc.

57. (1.) Let the time be x minutes past 3; then $x = 15 + \frac{x}{12}$, etc.

(2.) Let the time be x minutes past 6; then $x = 30 + \frac{x}{12}$, etc.

(3.) Let the time be x minutes past 9; then $x = 45 + \frac{x}{12}$, etc.

58. Let x be the number of apples; he pays for them $\frac{4x}{5}$ pence.

He sells x for $\frac{x}{2}$ pence, and x for $\frac{x}{3}$ pence.

Hence $\frac{x}{2} + \frac{x}{3} = \frac{4x}{5} + 1$; $x = 30$; $2x = 60$.

59. Let x be the number of sovereigns he had at first.

Then he has $\frac{x}{2} - \frac{1}{2}$ left after the first gift;

and he has $\frac{1}{2}\left(\frac{x}{2} - \frac{1}{2}\right) - \frac{1}{2}$ left after the second gift.

Then $\frac{1}{2}\left(\frac{x}{2} - \frac{1}{2}\right) - \frac{1}{2} = 0$; $\frac{x-1}{4} = \frac{1}{2}$; $x = 3$.

60. $\dfrac{\frac{2}{3} + n}{3n + \frac{69}{3}} = \frac{1}{33}$; $\frac{2 + 3n}{9n + 69} = \frac{1}{33}$; $66 + 99n = 9n + 69$, etc.

61. Let x be the number of days; then $x - 3$ is the number of days that the pursuing army is in motion; and $18x + 25 = 23(x - 3)$, etc.

62. Let x be his original income in pounds;

$$\text{then } x-\frac{x}{40}-\frac{1}{13}\left(x-\frac{x}{40}\right)=540, \text{ etc.}$$

63. Let x be the original sum; then first remainder is $\frac{x}{2}-50$, or, $\frac{x-100}{2}$;

second remainder is $\frac{4}{5}$ of $\frac{x-100}{2}-30$, or, $\frac{2x-200-150}{5}$;

third remainder is $\frac{3}{4}$ of $\frac{2x-350}{5}-20$, or, $\frac{6x-1050-400}{20}$;

thus $\frac{6x-1450}{20}=10$; $6x=1650$; $x=275$.

64. Let $2x$ be the number of eggs bought; for x I gave $\frac{x}{2}$ pence, and for x I gave $\frac{x}{3}$ pence; and I sold $2x$ eggs for $\frac{4x}{5}$ pence.

Thus $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+1$; $x=30$; $2x=60$.

65. Let x be the number of minutes in which C would fill it; then $\frac{1200}{x}$ = number of gall. let in by C each minute; and therefore $\frac{1200}{x}+10$ = number of gall. let in by A and B each minute.

Hence $\frac{1200}{x}+10+\frac{1200}{x}=\frac{1200}{24}$; whence $x=60$.

Thus A takes 90 minutes and C 60 minutes.

Also B lets in $\left(\frac{1200}{24}-\frac{1200}{90}-\frac{1200}{60}\right)$ gall. per minute, or,

$50-\frac{40}{3}-20$, or, $\frac{50}{3}$ gall. per minute.

$\therefore$ B takes $1200\div\frac{50}{3}$ minutes, or, 72 minutes to fill it.

66. Let x be the number of days B would take; then $2x$ is the number A would take.

Then $\frac{1}{x}+\frac{1}{2x}$ is the part A and B drink daily,

$\therefore$ $\frac{3}{4}$ of $\left(\frac{1}{x}+\frac{1}{2x}\right)$ is the part C drinks daily.

Then $\frac{1}{x}+\frac{1}{2x}+\frac{3}{4x}+\frac{3}{8x}=\frac{1}{24}$; $x=63$, etc.

67. Let x be the number of shots each fires.

Then $\frac{7x}{12}$ is the number of A's hits, and $\frac{9x}{12}$ of B's hits.

Thus $\frac{7x}{12}+\frac{9x}{12}=32$; $x=24$.

68. Let x be the number of horses ; then $2x$ is the number of oxen, and $100-3x$ is the number of sheep. Reducing all the prices to shillings,

$$440x+250\times 2x+30(100-3x)=4700\ ;\ x=2, \text{ etc.}$$

69. Let x be the whole number of marks.

Then $\frac{x}{5}=$ number of marks obtained by B.

And $\frac{2x}{5}=$number of marks assigned to book work in the paper.

Also $\frac{2x}{5}=$number of marks obtained by A for riders.

$\therefore$ $\frac{2x}{5}+\frac{2x}{5}=160$; $x=200$.

70. Since the man made a mistake of 55 minutes, the distance between the hands was 5 minutes.

Let x be the number of minutes past 2.

Then $x+5=10+\frac{x}{12}$; $x=5\frac{5}{11}$.

71. Let x be the original force ; then $\frac{5x}{6}-4000=$remainder after the defeat.

Hence $\frac{5x}{6}-1000=$ number after the reinforcement.

Then $\frac{3}{4}\left(\frac{5x}{6}-1000\right)=18000$; $\frac{5x}{8}-750=18000$; $x=30000$.

72. Let x be the original debt; then $x+\frac{x}{4}-25000000=$ debt after the peace.

Then $\frac{4\frac{1}{2}\times x}{100}=\frac{4\left(x+\frac{x}{4}-25000000\right)}{100}$; $\frac{9x}{2}=5x-100000000$, etc.

73. The influx goes on for 10 days and 15 hours, or, 255 hours.

The consumption extends over 10 days (excluding Sunday) of 15 hours each, or, 150 hours.

Hence if x be the number of gallons that come in each hour,

$$255x+2250=150\times 2x;\ x=50.$$

LX.

1. $a+x+\frac{3a}{x}=\frac{ax+x^2+3a}{x}$.

2. $\frac{a^2+ax}{x^2}-\frac{2x-2a}{x}=\frac{a^2+ax-2x^2+2ax}{x^2}=\frac{a^2+3ax-2x^2}{x^2}$.

3. $\frac{x-y}{x}+\frac{2y}{x-y}=\frac{x^2-2xy+y^2+2xy}{x(x-y)}=\frac{x^2+y^2}{x(x-y)}$.

4. $\frac{4a+4b}{a-b}-\frac{2a^2-2b^2}{a^2+b^2}=\frac{4a^3+4ab^2+4a^2b+4b^3-2a^3+2ab^2+2a^2b-2b^3}{(a-b)(a^2+b^2)}$, etc

LXI.

1. $\frac{5x+4}{35}-\frac{23x}{70}=\frac{10x+8-23x}{70}=\frac{8-13x}{70}$.

2. $\frac{\frac{x^2-y^2}{xy}}{x-y}=\frac{x^2-y^2}{xy(x-y)}=\frac{x+y}{xy}$.

3. $\dfrac{1-x^2}{\dfrac{x+1}{x}}=\dfrac{x(1-x^2)}{1+x}=x(1-x).$

4. $\dfrac{y\left(\dfrac{x+y}{y}\right)}{x\left(\dfrac{x-y}{x}\right)}=\dfrac{x+y}{x-y}.$

5. $\dfrac{5x^2+x^3+1}{x^2}\div\dfrac{2x^2-x^3+1}{x^2}=\dfrac{5x^2+x^3+1}{2x^2-x^3+1}.$

6. $\dfrac{x^3+1}{x^2}\div\dfrac{x+1}{x}=\dfrac{(x+1)(x^2-x+1)x}{x^2(x+1)}=\dfrac{x^2-x+1}{x}.$

7. $\dfrac{a^3-1}{a^2}\div\dfrac{a-1}{a}=\dfrac{(a-1)(a^2+a+1)a}{a^2(a-1)}=\dfrac{a^2+a+1}{a}.$

8. $\dfrac{x^2-ax+x^2+ax}{x^2-a^2}\div\dfrac{2x}{x^2-a^2}=\dfrac{2x^2}{2x}=x.$

9. $\dfrac{2x}{x^2+x^2}=\dfrac{2x}{2x^2}=\dfrac{1}{x}.$

10. $\dfrac{x^2}{1+x}+1-\dfrac{1}{1+x}=\dfrac{x^2+1+x-1}{1+x}=\dfrac{x(1+x)}{1+x}=x.$

11. $\dfrac{x^2-2xy+y^2+x^2+2xy+y^2}{x^2-y^2}\div\dfrac{x^2-2xy+y^2-x^2-2xy-y^2}{x^2-y^2}$

$=\dfrac{2x^2+2y^2}{-4xy}$, etc.

12. $\dfrac{1+x+x^2}{1}\div\dfrac{x^2+x+1}{x^2}=x^2.$

13. $\dfrac{a^2+2ab+b^2+b^2}{b(a+b)}\div\dfrac{a+b}{ab}=\dfrac{(a^2+2ab+2b^2)ab}{b(a+b)(a+b)}$, etc.

14. $\dfrac{2m^2-3m+1}{m}\div\dfrac{2m-1}{m}=\dfrac{2m^2-3m+1}{2m-1}=m-1.$

15. $\dfrac{c+b+a}{abc}\div\dfrac{(a+b+c)(a-b-c)}{ab}=\dfrac{ab(a+b+c)}{abc(a+b+c)(a-b-c)}$, etc.

LXII.

1. $\frac{a^4}{2a^4}+\frac{3a^3}{2a^4}+\frac{2a^2}{2a^4}+\frac{5a}{2a^4}=\frac{1}{2}+\frac{3}{2a}+\frac{1}{a^2}+\frac{5}{2a^3}$.

2. $\frac{a^2bc}{abcd}+\frac{ab^2d}{abcd}+\frac{abc^2}{abcd}+\frac{bcd^2}{abcd}=\frac{a}{d}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$.

3. $\frac{x^3}{x^2y^2}-\frac{3x^2y}{x^2y^2}+\frac{3xy^2}{x^2y^2}-\frac{y^3}{x^2y^2}=\frac{x}{y^2}-\frac{3}{y}+\frac{3}{x}-\frac{y}{x^2}$.

4. $\frac{9a^3}{108}-\frac{12a^2}{108}+\frac{6a}{108}-\frac{3}{108}=\frac{a^3}{12}-\frac{a^2}{9}+\frac{a}{18}-\frac{1}{36}$.

5. $\frac{18p^2}{3pqrs}+\frac{12q^2}{3pqrs}-\frac{36r^2}{3pqrs}+\frac{72s^2}{3pqrs}=\frac{6p}{qrs}+\frac{4q}{prs}-\frac{12r}{pqs}+\frac{24s}{pqr}$.

6. $\frac{10x^3}{1000}-\frac{25x^2}{1000}+\frac{75x}{1000}-\frac{125}{1000}=\frac{x^3}{100}-\frac{x^2}{40}+\frac{3x}{40}-\frac{1}{8}$.

LXIII.

1.
$$\begin{array}{l}
1+a)\,2\,(2-2a+2a^2-2a^3+2a^4\ldots \\
\qquad\quad \underline{2+2a} \\
\qquad\qquad -2a \\
\qquad\qquad \underline{-2a-2a^2} \\
\qquad\qquad\qquad\quad 2a^2 \\
\qquad\qquad\qquad\quad \underline{2a^2+2a^3} \\
\qquad\qquad\qquad\qquad\quad -2a^3 \\
\qquad\qquad\qquad\qquad\quad \underline{-2a^3-2a^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2a^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{2a^4+2a^5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2a^5
\end{array}$$

2. $m+2)\,m\,(1-\frac{2}{m}+\frac{4}{m^2}-\frac{8}{m^3}+\frac{16}{m^4}\cdots$

$$\begin{array}{l}
\underline{m+2}\\
-2\\
\underline{-2-\frac{4}{m}}\\
\frac{4}{m}\\
\underline{\frac{4}{m}+\frac{8}{m^2}}\\
-\frac{8}{m^2}\\
\underline{-\frac{8}{m^2}-\frac{16}{m^3}}\\
\frac{16}{m^3}
\end{array}$$

3. $a+b)\,a-b\,(1-\frac{2b}{a}+\frac{2b^2}{a^2}-\frac{2b^3}{a^3}+\frac{2b^4}{a^4}\cdots$

$$\begin{array}{l}
\underline{a+b}\\
-2b\\
\underline{-2b-\frac{2b^2}{a}}\\
\frac{2b^2}{a}\\
\underline{\frac{2b^2}{a}+\frac{2b^3}{a^2}}\\
-\frac{2b^3}{a^2}\\
\underline{-\frac{2b^3}{a^2}-\frac{2b^4}{a^3}}\\
\frac{2b^4}{a^3}
\end{array}$$

4. $a^2-x^2)\,a^2+x^2\,(1+\frac{2x^2}{a^2}+\frac{2x^4}{a^4}+\frac{2x^6}{a^6}+\frac{2x^8}{a^8}\cdots$

$$\begin{array}{l}
\underline{a^2-x^2}\\
2x^2\\
\underline{2x^2-\frac{2x^4}{a^2}}\\
\frac{2x^4}{a^2}\\
\underline{\frac{2x^4}{a^2}-\frac{2x^6}{a^4}}\\
\frac{2x^6}{a^4}\\
\underline{\frac{2x^6}{a^4}-\frac{2x^8}{a^6}}\\
\frac{2x^8}{a^6}
\end{array}$$

5. $a-x)\,ax\,(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\frac{x^5}{a^4}\cdots$

$$\begin{array}{l}
\underline{ax-x^2}\\
x^2\\
\underline{x^2-\frac{x^3}{a}}\\
\frac{x^3}{a}\\
\underline{\frac{x^3}{a}-\frac{x^4}{a^2}}\\
\frac{x^4}{a^2}\\
\underline{\frac{x^4}{a^2}-\frac{x^5}{a^3}}\\
\frac{x^5}{a^3}
\end{array}$$

6. $a+x)\,b\,(\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}\ldots$

$$\begin{array}{l}
b+\frac{bx}{a} \\
\hline
-\frac{bx}{a} \\
-\frac{bx}{a}-\frac{bx^2}{a^2} \\
\hline
\frac{bx^2}{a^2} \\
\frac{bx^2}{a^2}-\frac{bx^3}{a^3} \\
\hline
\frac{bx^3}{a^3} \\
\frac{bx^3}{a^3}-\frac{bx^4}{a^4} \\
\hline
\frac{bx^4}{a^4}
\end{array}$$

7. $1+2x-2x^2)\,1\,(1-2x+6x^2-16x^3+44x^4\ldots$

$$\begin{array}{r}
1+2x-2x^2 \\
\hline
-2x+2x^2 \\
-2x-4x^2+4x^3 \\
\hline
6x^2-4x^3 \\
6x^2+12x^3-12x^4 \\
\hline
-16x^3+12x^4 \\
-16x^3-32x^4+32x^5 \\
\hline
-44x^4-32x^5
\end{array}$$

8. $1-x+x^2)\,1+x\,(1+2x+x^2-x^3-2x^4\ldots$

$$\begin{array}{r}
1-x+x^2 \\
\hline
2x-x^2 \\
2x-2x^2+2x^3 \\
\hline
x^2-2x^3 \\
x^2-x^3+x^4 \\
\hline
-x^3-x^4 \\
-x^3+x^4-x^5 \\
\hline
-2x^4+x^5
\end{array}$$

9. $1-2b)\,1+b\,(1+3b+6b^2+12b^3+24b^4\ldots$

$$\begin{array}{r}
1-2b \\
\hline
3b \\
3b-6b^2 \\
\hline
6b^2 \\
6b^2-12b^3 \\
\hline
12b^3 \\
12b^3-24b^4 \\
\hline
24b^4
\end{array}$$

10. $x+b)x^3-b^3(x^2-bx+b^2-\frac{2b^3}{x}+\frac{2b^4}{x^2}\cdots$

$$x^3+bx^2$$
$$-bx^2-b^3$$
$$-bx^2-b^2x$$
$$b^2x-b^3$$
$$b^2x+b^3$$
$$-2b^3$$
$$-2b^3-\frac{2b^4}{x}$$
$$\frac{2b^4}{x}$$

11. $x-b)a^2(\frac{a^2}{x}+\frac{a^2b}{x^2}+\frac{a^2b^2}{x^3}+\frac{a^2b^3}{x^4}+\frac{a^2b^4}{x^5}\cdots$

$$a^2-\frac{a^2b}{x}$$
$$\frac{a^2b}{x}$$
$$\frac{a^2b}{x}-\frac{a^2b^2}{x^2}$$
$$\frac{a^2b^2}{x^2}$$
$$\frac{a^2b^2}{x^2}-\frac{a^2b^3}{x^3}$$
$$\frac{a^2b^3}{x^3}$$
$$\frac{a^2b^3}{x^3}-\frac{a^2b^4}{x^4}$$
$$\frac{a^2b^4}{x^4}$$

12. $a^2+2ax+x^2)a^2(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\frac{5x^4}{a^4}\cdots$

$$a^2+2ax+x^2$$
$$-2ax-x^2$$
$$-2ax-4x^2-\frac{2x^3}{a}$$
$$3x^2+\frac{2x^3}{a}$$
$$3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}$$
$$-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$
$$-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}$$
$$\frac{5x^4}{a^2}+\frac{4x^5}{a^3}$$

13. Dividend $=$ (Divisor and Quotient) $+$ Remainder

$$=(x-a)(x^2-2ax)+4a^3=x^3-3ax^2+2a^2x+4a^3.$$

14. Dividend $=(m-5)(m^3+5m^2+15m+34)+75$

$$=m^4-10m^2-41m-95.$$

LXIV.

1. $\dfrac{10x^2+15x+6}{30}\times\dfrac{4x+3}{12}=\dfrac{40x^3+90x^2+69x+18}{360}$, etc.

2. $\dfrac{6a^2-5a+10}{30}\times\dfrac{5a-4}{20}=\dfrac{30a^3-49a^2+70a-40}{600}$, etc.

3.
$$\begin{array}{l}
x^3+x+\dfrac{1}{x}+\dfrac{1}{x^3}\\
x-\dfrac{1}{x}\\
\hline
x^4+x^2+1+\dfrac{1}{x^2}\\
\qquad -x^2-1-\dfrac{1}{x^2}-\dfrac{1}{x^4}\\
\hline
x^4 \qquad\qquad\qquad\quad -\dfrac{1}{x^4}
\end{array}$$

4.
$$\begin{array}{l}
x^2-1+\dfrac{1}{x^2}\\
x^2+1+\dfrac{1}{x^2}\\
\hline
x^4-x^2+1\\
\qquad +x^2-1+\dfrac{1}{x^2}\\
\qquad\qquad +1-\dfrac{1}{x^2}+\dfrac{1}{x^4}\\
\hline
x^4 \qquad +1 \qquad\quad +\dfrac{1}{x^4}
\end{array}$$

5. $\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)=\dfrac{1}{a^4}-\dfrac{1}{b^4}.$

6. $\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{ab}-\dfrac{1}{b^2}+\dfrac{1}{bc}+\dfrac{1}{ac}-\dfrac{1}{bc}+\dfrac{1}{c^2}=\dfrac{1}{a^2}+\dfrac{2}{ac}-\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

7. $1+\dfrac{b}{a}+\dfrac{b^2}{a^2}-\dfrac{b}{a}-\dfrac{b^2}{a^2}-\dfrac{b^3}{a^3}+\dfrac{b^2}{a^2}+\dfrac{b^3}{a^3}+\dfrac{b^4}{a^4}=1+\dfrac{b^2}{a^2}+\dfrac{b^4}{a^4}.$

8. $$\begin{array}{l} 1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\ 1+\frac{1}{2}x+\frac{1}{4}x^2 \\ \hline 1-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{16}x^3 \\ \quad +\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{16}x^3-\frac{1}{32}x^4 \\ \qquad\quad +\frac{1}{4}x^2-\frac{1}{8}x^3+\frac{1}{32}x^4-\frac{1}{64}x^5 \\ \hline 1+\frac{x^2}{8} \qquad\quad -\frac{x^3}{8} \quad -\frac{x^5}{64}. \end{array}$$

9. $$\frac{15+18x-14x^2}{6x^2}\times\frac{4-2x-x^2}{2x^2}=\frac{60+42x-107x^2+10x^3+14x^4}{12x^4}\text{, etc.}$$

10. $$\frac{a^4}{b^4}+1+\frac{2a^2}{b^2}-1-\frac{b^4}{a^4}-\frac{2b^2}{a^2}-\frac{2a^2}{b^2}-\frac{2b^2}{a^2}-4=\frac{a^4}{b^4}-\frac{b^4}{a^4}-\frac{4b^2}{a^2}-4.$$

LXV.

1. $$\frac{x^4-1}{x^2}\div\frac{x^2+1}{x}=\frac{x^2-1}{x}\text{, etc.}$$

2. $$\frac{a^2b^2-1}{b^2}\div\frac{ab-1}{b}=\frac{ab+1}{b}\text{, etc.}$$

3. $$\frac{m^3n^3+1}{n^3}\div\frac{mn+1}{n}=\frac{m^2n^2-mn+1}{n^2}\text{, etc.}$$

4. $$\frac{c^5d^5-1}{d^5}\div\frac{cd-1}{d}=\frac{c^4d^4+c^3d^3+c^2d^2+cd+1}{d^4}\text{, etc.}$$

5. $$\frac{x^4+2x^2y^2+y^4}{x^2y^2}\div\frac{x^2+y^2}{xy}=\frac{x^2+y^2}{xy}\text{, etc.}$$

6. $$\frac{b^4+a^2b^2+a^4}{a^4b^4}\div\frac{b^2-ab+a^2}{a^2b^2}=\frac{b^2+ab+a^2}{a^2b^2}\text{, etc.}$$

7. $\dfrac{x^6-y^6-3x^4y^2+3x^2y^4}{x^3y^3} \div \dfrac{x^2-y^2}{xy} = \dfrac{x^4-2x^2y^2+y^4}{x^2y^2}$, etc.

8. $\dfrac{6x^5-32x^4+77x^3-86x^2-66x+216}{8} \div \dfrac{x^2-2x+6}{2}$

$= \dfrac{6x^3-10x^2+16x+36}{4}$, etc.

9. $\dfrac{a^6+b^6}{a^3b^3} \div \dfrac{a^2+b^2}{ab} = \dfrac{a^4-a^2b^2+b^4}{a^2b^2}$, etc.

10. $\dfrac{b^3c^3+a^3c^3+a^3b^3-3a^2b^2c^2}{a^3b^3c^3} \div \dfrac{bc+ac+ab}{abc}$

$= \dfrac{b^2c^2-abc^2-ab^2c+a^2c^2-a^2bc+a^2b^2}{a^2b^2c^2}$, etc.

LXVI.

1. $\dfrac{x-3}{10} \times \dfrac{50x+7}{100} = \dfrac{50x^2-143x-21}{1000}$, etc.

2. $\left(\dfrac{5x}{100}+7\right) \times \left(\dfrac{2x}{10}-3\right) = \dfrac{5x+700}{100} \times \dfrac{2x-30}{10} = \dfrac{10x^2+1250x-21000}{1000}$, etc.

3. $\dfrac{3x-2y}{10} \times \dfrac{4x+7y}{10} = \dfrac{12x^2+13xy-14y^2}{100}$, etc.

4. $\dfrac{43x+52y}{10} \times \dfrac{4x-6y}{100} = \dfrac{172x^2-50xy-312y^2}{1000}$, etc.

5. $\cdot000027 - \cdot001 + \cdot000343 + \cdot00063 = \cdot001 - \cdot001 = 0.$

6. $\cdot343 - \cdot0441 + \cdot00189 - \cdot000027 = \cdot34489 - \cdot044127 = \cdot300763.$

LXVII.

1. $a_1x\left(\dfrac{a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots}{a_1x}\right) = a\,x\left(1+\dfrac{a_2x}{a_1}+\dfrac{a_3x^2}{a_1}+\dfrac{a_4x^3}{a_1}+\ldots\right).$

2. $$xyz\left(\frac{xy - xz + yz}{xyz}\right) = xyz\left(\frac{1}{z} - \frac{1}{y} + \frac{1}{x}\right).$$

3. $$x^2\left(\frac{x^2 + xy + y^2}{x^2}\right) = x^2\left(1 + \frac{y}{x} + \frac{y^2}{x^2}\right).$$

4. $$(a+b)\left\{\frac{(a+b)^3 - c(a+b)^2 - d(a+b) + e}{a+b}\right\}$$
$$= (a+b)\left\{(a+b)^2 - c(a+b) - d + \frac{e}{a+b}\right\}.$$

LXVIII.

1. Let $\frac{a}{b} = \lambda$; then $\frac{c}{d} = \lambda$; and $a = \lambda b$, $c = \lambda d$. Then

(1.) $$\frac{a-b}{b} = \frac{\lambda b - b}{b} = \lambda - 1$$
$$\frac{c-d}{d} = \frac{\lambda d - d}{d} = \lambda - 1.$$

(2.) $$\frac{a}{a+b} = \frac{\lambda b}{\lambda b + b} = \frac{\lambda}{\lambda + 1}$$
$$\frac{c}{c+d} = \frac{\lambda d}{\lambda d + d} = \frac{\lambda}{\lambda + 1}.$$

(3.) $$\frac{3a}{4a - 5b} = \frac{3\lambda b}{4\lambda b - 5b} = \frac{3\lambda}{4\lambda - 5}$$
$$\frac{3c}{4c - 5d} = \frac{3\lambda d}{4\lambda d - 5d} = \frac{3\lambda}{4\lambda - 5}.$$

(4.) $$\frac{a^2 + b^2}{a^2 - b^2} = \frac{\lambda^2 b^2 + b^2}{\lambda^2 b^2 - b^2} = \frac{\lambda^2 + 1}{\lambda^2 - 1}$$
$$\frac{c^2 + d^2}{c^2 - d^2} = \frac{\lambda^2 d^2 + d^2}{\lambda^2 d^2 - d^2} = \frac{\lambda^2 + 1}{\lambda^2 - 1}.$$

(5.) $$\frac{8a + b}{4a + 7b} = \frac{8\lambda b + b}{4\lambda b + 7b} = \frac{8\lambda + 1}{4\lambda + 7}$$
$$\frac{8c + d}{4c + 7d} = \frac{8\lambda d + d}{4\lambda d + 7d} = \frac{8\lambda + 1}{4\lambda + 7}.$$

(6.) $\frac{a^2-b^2}{c^2-d^2}=\frac{\lambda^2b^2-b^2}{\lambda^2d^2-d^2}=\frac{b^2(\lambda^2-1)}{d^2(\lambda^2-1)}=\frac{b^2}{d^2}$

$\frac{ab}{cd}=\frac{\lambda b\times b}{\lambda d\times d}=\frac{b^2}{d^2}.$

(7.) $\frac{11a+b}{11c+d}=\frac{11\lambda b+b}{11\lambda d+d}=\frac{b(11\lambda+1)}{d(11\lambda+1)}=\frac{b}{d}$

$\frac{13a+b}{13c+d}=\frac{13\lambda b+b}{13\lambda d+d}=\frac{b(13\lambda+1)}{d(13\lambda+1)}=\frac{b}{d}.$

(8.) $\frac{a^2-ab+b^2}{a^2+ab+b^2}=\frac{\lambda^2b^2-\lambda b^2+b^2}{\lambda^2b^2+\lambda b^2+b^2}=\frac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}$

$\frac{c^2-cd+d^2}{c^2+cd+d^2}=\frac{\lambda^2d^2-\lambda d^2+d^2}{\lambda^2d^2+\lambda d^2+d^2}=\frac{\lambda^2-\lambda+1}{\lambda^2+\lambda+1}.$

2. Let each of the fractions $=\lambda$

Then $l=a\lambda-b\lambda$; $m=b\lambda-c\lambda$; $n=c\lambda-a\lambda$

And $l+m+n=a\lambda-b\lambda+b\lambda-c\lambda+c\lambda-a\lambda=0.$

3. Let each of the fractions $=\lambda$

Then $a=\lambda b$, $c=\lambda d$, $e=\lambda f$

And $\frac{la+mc+ne}{lb+md+nf}=\frac{l\lambda b+m\lambda d+n\lambda f}{lb+md+nf}=\lambda=\frac{a}{b}.$

4. Let each of the fractions $=\lambda$

Then $a+b=\lambda b$
$b+c=\lambda c$
$c+a=\lambda a$

Adding, $2a+2b+2c=\lambda(a+b+c)$
Hence $\lambda=2$, and $\therefore\ a+b=2b$, or $a=b$
Similarly we can show that $a=c$, and $b=a$.

5. Let each of the fractions $=\lambda$

Then $a_1=\lambda b_1$; $a_2=\lambda b_2$; $a_3=\lambda b_3$

And $\frac{2a_1+3a_2+4a_3}{2b_1+3b_2+4b_3}=\frac{2\lambda b_1+3\lambda b_2+4\lambda b_3}{2b_1+3b_2+4b_3}=\lambda=\frac{a_1}{b_1}.$

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}$ is less than λ; and $\frac{e}{f}$ is less than λ

$\therefore a=\lambda b$; c is less than λd; e is less than λf

$\therefore a+c+e$ is less than $\lambda b+\lambda d+\lambda f$

$\therefore \frac{a+c+e}{b+d+f}$ is less than λ, that is, less than $\frac{a}{b}$

Next let $\frac{e}{f}=\mu$; then $\frac{a}{b}$ and $\frac{c}{d}$ are each greater than μ

$\therefore e=\mu f$; a is greater than μb; c is greater than μd

$\therefore a+c+e$ is greater than $\mu b+\mu d+\mu f$

$\therefore \frac{a+c+e}{b+d+f}$ is greater than μ, that is than $\frac{e}{f}$.

7. Let each fraction $=\lambda$; then $x_1=\lambda y_1$; and $x_2=\lambda y_2$

Then $\frac{4x_1+5y_1}{7x_1+9y_1}=\frac{4\lambda y_1+5y_1}{7\lambda y_1+9y_1}=\frac{4\lambda+5}{7\lambda+9}$

And $\frac{4x_2+5y_2}{7x_2+9y_2}=\frac{4\lambda y_2+5y_2}{7\lambda y_2+9y_2}=\frac{4\lambda+5}{7\lambda+9}$.

8. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{a^2+ab}{c^2+cd}=\frac{\lambda^2b^2+\lambda b^2}{\lambda^2d^2+\lambda d^2}=\frac{b^2(\lambda^2+\lambda)}{d^2(\lambda^2+\lambda)}=\frac{b^2}{d^2}$

And $\frac{ab-b^2}{cd-d^2}=\frac{\lambda b^2-b^2}{\lambda d^2-d^2}=\frac{b^2(\lambda-1)}{d^2(\lambda-1)}=\frac{b^2}{d^2}$.

9. Let each fraction $=\lambda$; then $a=\lambda b$; and $c=\lambda d$

Then $\frac{7a+b}{3a+5b}=\frac{7\lambda b+b}{3\lambda b+5b}=\frac{7\lambda+1}{3\lambda+5}$

And $\frac{7c+d}{3c+5d}=\frac{7\lambda d+d}{3\lambda d+5d}=\frac{7\lambda+1}{3\lambda+5}$.

10. Since b is greater than a

bc is greater than ac

$ab+bc$ is greater than $ab+ac$

$\frac{ab+bc}{b(b+c)}$ is greater than $\frac{ab+ac}{b(b+c)}$

$\frac{b(a+c)}{b(b+c)}$ is greater than $\frac{a(b+c)}{b(b+c)}$

$\frac{a+c}{b+c}$ is greater than $\frac{a}{b}$.

11. Since b is less than a, we can show, as in the preceding solution that $\frac{a+c}{b+c}$ is less than $\frac{a}{b}$.

LXIX.

1. $$3\times16+\frac{2\times4\times\frac{1}{4}}{1}-\frac{1}{\frac{1}{4}}=48+2-4=46.$$

2. $$\begin{array}{l}
7x^2-12x+5)\,14x^3+\;7x^2-56x+35\,(2x\\
\qquad\qquad\quad\;\underline{14x^3-24x^2+10x}\\
\qquad\qquad\qquad\quad\;\;31x^2-66x+35\\
\qquad\qquad\qquad\qquad\;\underline{7}\\
\qquad\qquad\qquad\quad\;217x^2-462x+245\,(31\\
\qquad\qquad\qquad\quad\;\underline{217x^2-372x+155}\\
\qquad\qquad\qquad\qquad\qquad\;\;-90x+90
\end{array}$$

Hence H. C. F. is $x-1$, and dividing both terms of the first fraction by it, we get $\frac{2x^2+3x-5}{7x-5}$

$$\begin{array}{l}
a^2+4a-45)\,a^3-39a+70\,(a-4\\
\qquad\qquad\quad\;\;\underline{a^3+4a^2-45a}\\
\qquad\qquad\qquad\;-4a^2+6a+70\\
\qquad\qquad\qquad\;\underline{-4a^2-16a+180}\\
\qquad\qquad\qquad\qquad\quad\;22a-110
\end{array}$$

Hence we find $a-5$ to be the H. C. F., and dividing both terms of the fraction by it, we get $\frac{a^2+5a-14}{a+9}$.

3. $$\frac{a^2+2ap+p^2-a^2+2ap-p^2}{a^2-p^2}\div\frac{a^2+2ap+p^2+a^2-2ap+p^2}{a^2-p^2}$$

$$=\frac{4ap}{2a^2+2p^2}=\frac{2ap}{a^2+p^2}.$$

4. $$\frac{6x^2-4y^2+3z^2}{24}+\frac{6y^2-4z^2+3x^2}{24}+\frac{6z^2+4x^2+3y^2}{24}=\frac{13x^2+5y^2+5z^2}{24}$$

$$\frac{13x^2+5y^2+5z^2}{24}-\frac{24z^2-24x^2+12y^2}{24}=\frac{37x^2-7y^2-19z^2}{24}.$$

5. $$\frac{16+\frac{1}{4}-1+4}{16-\frac{1}{4}-1+1}=\frac{19\frac{1}{4}}{15\frac{3}{4}}=\frac{77}{63}=\frac{11}{9}.$$

6. $$\frac{15x^2+18ax-14a^2}{6}\times\frac{4x^2-2ax-a^2}{2}$$

$$=\frac{60x^4+42ax^3-107a^2x^2+10a^3x+14a^4}{12}.$$

7. $$\frac{(a-b)(a^2+ab+b^2)}{(a-b)^2}=\frac{a^2+ab+b^2}{a-b}=a+2b+\frac{3b^2}{a-b},\text{ by division.}$$

8. $$\frac{x^2-2xy+y^2+2xy}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}=\frac{x^2+y^2}{x(x-y)}+\frac{y^3-xy^2}{x(x^2-y^2)}$$

$$=\frac{x^3+xy^2+x^2y+y^3+y^3-xy^2}{x(x^2-y^2)}=\frac{x^3+x^2y+2y^3}{x(x^2-y^2)}.$$

9. $$\begin{array}{rl}
5x^2+9x-2) & 60x^3-17x^2-4x+1\;(12x-25 \\
 & \underline{60x^3+108x^2-24x} \\
 & \qquad -125x^2+20x+1 \\
 & \qquad \underline{-125x^2-225x+50} \\
 & \qquad\qquad\qquad 245x-49
\end{array}$$

and $$\frac{245x-49}{5x^2+9x-2}=\frac{49(5x-1)}{(x+2)(5x-1)}=\frac{49}{x+2}.$$

10. $$\begin{array}{rl}
x^4-9x^3+7x^2+9x-8) & x^4+7x^3-9x^2-7x+8\;(1 \\
 & \underline{x^4-9x^3+7x^2+9x-8} \\
 & \qquad 16x^3-16x^2-16x+16
\end{array}$$

Divide by 16; $x^3-x^2-x+1)\,x^4-9x^3+7x^2+9x-8\,(x-8$

$$\begin{array}{r}
x^4-x^3-x^2+x \\
\hline
-8x^3+8x^2+8x-8 \\
-8x^3+8x^2+8x-8 \\
\hline
\end{array}$$

Hence x^3-x^2-x+1 is the H. C. F.

11. $$\frac{x^3}{x^4-1}+\frac{1}{1-x-\dfrac{1-x}{1-x^2-1}}=\frac{x^3}{x^4-1}+\frac{-x^2}{-x^2+x^3-1+x}$$

$$=\frac{x^3}{x^4-1}+\frac{-x^2(x+1)}{x^4-1}=\frac{-x^2}{x^4-1}=\frac{x^2}{1-x^4}.$$

12. $$a+ab+b^2\left(a+ab+\frac{ab^2}{1-b}\right)=a+ab+b^2\left(\frac{a+ab-ab-ab^2+ab^2}{1-b}\right)$$

$$=a+ab+\frac{ab^2}{1-b}=\frac{a+ab-ab-ab^2+ab^2}{1-b}=\frac{a}{1-b}.$$

13. $$\left(l+\frac{1}{l}\right)\left(l-\frac{1}{l}\right)\left(l^2+\frac{1}{l^2}\right)=\left(l^2-\frac{1}{l^2}\right)\left(l^2+\frac{1}{l^2}\right)=l^4-\frac{1}{l^4}.$$

14. $$\frac{a+b+2}{ab+a+b+1}+\frac{1}{c+1}=\frac{ac+bc+2c+a+b+2+ab+a+b+1}{abc+ab+ac+a+bc+b+c+1}$$

$$=\frac{ab+ac+bc+2a+2b+2c+3}{abc+ab+ac+a+bc+b+c+1}$$

If this fraction $=1$, then $ab+ac+bc+2a+2b+2c+3$

$=abc+ab+ac+a+bc+b+c+1$;

or, $ab+ac+bc+2a+2b+2c+3-ab-ac-a-bc-b-c-1=abc$,

or, $a+b+c+2=abc$.

15. $\dfrac{ax^3-a^2x^2-abx^2-b^2x^2+a^2bx+a^2b^2}{a^2x^2} \div x-a$

$$\begin{array}{l}
x-a)\;ax^3-a^2x^2-abx^2-b^2x^2+a^2bx+a^2b^2\;(ax^2-abx-b^2x-ab^2 \\
\qquad\underline{ax^3-a^2x^2\qquad\qquad} \\
\qquad\qquad\qquad -abx^2-b^2x^2+a^2bx \\
\qquad\qquad\qquad \underline{-abx^2+a^2bx\qquad\quad} \\
\qquad\qquad\qquad\qquad\quad -b^2x^2+a^2b^2 \\
\qquad\qquad\qquad\qquad\quad \underline{-b^2x^2+ab^2x} \\
\qquad\qquad\qquad\qquad\qquad\qquad -ab^2x+a^2b^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-ab^2x+a^2b^2}
\end{array}$$

Hence the result is $\dfrac{ax^2-abx-b^2x-ab^2}{a^2x^2}$, or, $\dfrac{1}{a}-\dfrac{b}{ax}-\dfrac{b^2}{a^2x}-\dfrac{b^2}{ax^2}$.

16. $\left(\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}\right) \div \left(\dfrac{b}{ac}+\dfrac{c}{ab}+\dfrac{a}{bc}\right)=1$

Also, since $s=\dfrac{a+b+c}{2}$

$s-a=\dfrac{b+c-a}{2}$; $s-b=\dfrac{a+c-b}{2}$; $s-c=\dfrac{a+b-c}{2}$.

Hence $\dfrac{s.(s-a)\ +\ (s-b)\ (s-c)}{bc}=\dfrac{\dfrac{(b+c)^2-a^2}{4}+\dfrac{a^2-(b-c)^2}{4}}{bc}$

$=\dfrac{b^2+2bc+c^2-a^2+a^2-b^2+2bc-c^2}{4bc}=\dfrac{4bc}{4bc}=1.$

17. $\dfrac{1}{1+\dfrac{x}{a}}+\dfrac{1}{1-\dfrac{x}{a}}+\dfrac{2}{1+\dfrac{x^2}{a^2}}=\dfrac{a}{a+x}+\dfrac{a}{a-x}+\dfrac{2a^2}{a^2+x^2}$

$=\dfrac{a^2-ax+a^2+ax}{a^2-x^2}+\dfrac{2a^2}{a^2+x^2}=\dfrac{2a^2(a^2+x^2)\ +2a^2\,(a^2-x^2)}{a^4-x^4}=\dfrac{4a^4}{a^4-x^4}$

18. $$\frac{a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}=\frac{2(a^2+b^2)}{a^2-b^2}-\frac{2(a^2-b^2)}{a^2+b^2}$$

$$=\frac{2(a^4+2a^2b^2+b^4)-2(a^4-2a^2b^2+b^4)}{a^4-b^4}=\frac{8a^2b^2}{a^4-b^4}.$$

19. $$\frac{2ab-a^2-2ab-b^2}{2a(a+b)}+\frac{a^2+b^2}{2a(a-b)}=\frac{(a-b)(-a^2-b^2)+(a+b)(a^2+b^2)}{2a(a^2-b^2)}$$

$$=\frac{-a^3-ab^2+a^2b+b^3+a^3+ab^2+a^2b+b^3}{2a(a^2-b^2)}=\frac{2a^2b+2b^3}{2a(a^2-b^2)}=\frac{b(a^2+b^2)}{a(a^2-b^2)}.$$

20. $$\frac{a^2-ab+b^2}{(a-b)^3}\times\frac{(a+b)(a-b)}{a^2+b^2}=\frac{(a^2-ab+b^2)(a+b)}{(a-b)^2(a^2+b^2)}=\frac{a^3+b^3}{(a-b)^2(a^2+b^2)}.$$

21. $$\frac{2}{(x^2-1)^2}-\frac{1}{2(x-1)^2}+\frac{1}{x^2-1}=\frac{2}{(x^2-1)^2}-\frac{(x+1)^2}{2(x^2-1)^2}+\frac{x^2-1}{(x^2-1)^2}$$

$$=\frac{4-x^2-2x-1+2x^2-2}{2(x^2-1)^2}=\frac{x^2-2x+1}{2(x^2-1)^2}=\frac{1}{2(x+1)^2}.$$

22. $$\frac{(a+b)^2-c^2}{c^2-(a-b)^2}\div\frac{a+b+c}{b+c-a}=\frac{(a+b+c)(a+b-c)(b+c-a)}{(c+a-b)(c-a+b)(a+b+c)}=\frac{a+b-c}{a-b+c}.$$

23. $$\left(\frac{x^2}{x+1}+1-\frac{1}{x+1}\right)\div\left(\frac{x^2}{x-1}-x-\frac{1}{x-1}\right)$$

$$=\frac{x^2+x+1-1}{x+1}\div\frac{x^2-x^2+x-1}{x-1}=\frac{x^2+x}{x+1}\div 1=\frac{x(x+1)}{x+1}=x.$$

24. $$\left(\frac{\frac{a+b}{2}-a}{\frac{a+b}{2}-b}\right)^3-\frac{\frac{a+b}{2}-2a+b}{\frac{a+b}{2}+a-2b}=\left(\frac{b-a}{a-b}\right)^3-\frac{3b-3a}{3a-3b}=(-1)^3-(-1)$$

$$=-1+1=0.$$

25. $$\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}+\frac{(b+a-c)(b-a+c)}{(a+b+c)(a+b-c)}+\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}$$

$$=\frac{a+b-c}{a+c+b}+\frac{b-a+c}{a+b+c}+\frac{c+a-b}{b+c+a}=\frac{b+c+a}{a+b+c}=1.$$

26. $$\frac{x(x-4)(x^2-4)(x^2-4)}{x^2(x-2)(x-2)}=\frac{(x-4)(x+2)(x+2)}{x}=\frac{(x-4)(x+2)^2}{x}.$$

27. $$\frac{(a+1)(a-1)(a^3+1)(a^3-1)}{(a+1)(a+1)a^2(a-1)(a-1)}=\frac{(a^2-a+1)(a^2+a+1)}{a^2}=\frac{a^4+a^2+1}{a^2}.$$

28. $$\frac{1+x-x^2}{x^3}-\frac{1}{(x^2+1)^2}+\frac{(x^2+1)(x-1)}{(x^2+1)^2}-\frac{3}{x^2(x^2+1)^2}$$

$$=\frac{1+x-x^2}{x^3}-\frac{x^3}{x^3(x^2+1)^2}+\frac{x^3(x^3-x^2+x-1)}{x^3(x^2+1)^2}-\frac{3x}{x^3(x^2+1)^2}$$

$$=\frac{(x^4+2x^2+1)(1+x-x^2)-x^3+x^3(x^3-x^2+x-1)-3x}{x^3(x^2+1)^2}$$

$$=\frac{1+x+x^2+2x^3-x^4+x^5-x^6-x^3+x^6-x^5+x^4-x^3-3x}{x^3(x^2+1)^2}$$

$$=\frac{1-2x+x^2}{x^3(x^2+1)^2}=\frac{(x-1)^2}{x^3(x^2+1)^2}.$$

29. $$\frac{x^6-a^2x^4+a^4x^2-a^6}{a^3x^3}\div\frac{x^2-a^2}{ax}=\frac{x^4+a^4}{a^2x^2}=\frac{x^2}{a^2}+\frac{a^2}{x^2}.$$

30. $$\left\{\frac{a^2+2ab+b^2-a^2+2ab-b^2}{2(a^2-b^2)}+\frac{2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}$$

$$=\left\{\frac{2ab+2b^2}{a^2-b^2}\right\}\frac{a-b}{2b}=\frac{2b(a+b)(a-b)}{(a+b)(a-b)2b}=1.$$

31. $$\frac{a^2+b^2+c^2+2ab+2ac+2bc+b^2-2bc+c^2+c^2-2ac+a^2-2ab+b^2}{a^2+b^2+c^2}$$

$$=\frac{3a^2+3b^2+3c^2}{a^2+b^2+c^2}=3.$$

32. $$\frac{1+3x^2+2x^3}{(3-2x-7x^2)^4}-\frac{(1-x-3x^2)(3-2x-7x^2)}{(3-2x-7x^2)^4}$$

$$=\frac{1+3x^2+2x^3-(3-5x-14x^2+13x^3+21x^4)}{(3-2x-7x^2)^4}$$

$$=\frac{-2+5x+17x^2-11x^3-21x^4}{(3-2x-7x^2)^4}.$$

33. $$\frac{x^4+2x^2y^2+y^4-x^4+2x^2y^2-y^4}{(x^2-y^2)(x^2+y^2)}\div\frac{x^2+2xy+y^2-x^2+2xy-y^2}{(x-y)(x+y)}$$

$$=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\times\frac{(x-y)(x+y)}{4xy}=\frac{xy}{x^2+y^2}.$$

34. $$\frac{x^2-y^2}{y^2}\times\frac{y}{x-y}+\frac{x^3-y^3}{y^3}\times\frac{-y^2}{x^2+xy+y^2}=\frac{x+y}{y}+\frac{(x-y)(-1)}{y}=\frac{2y}{y}=2.$$

35. $$\frac{a(a-b)}{(a-b)(a^2+ab+b^2)}\times\frac{a^2+ab+b^2}{a+b}+\frac{a^3-b^3}{a^3+b^3}\times\frac{a^2-ab+b^2}{a^2+ab+b^2}$$

$$=\frac{a}{a+b}+\frac{a-b}{a+b}=\frac{2a-b}{a+b}.$$

36. $$\frac{2-(x-1)}{4(x-1)^2}+\frac{(x-1)^2-4}{4(x-1)^2(x+1)}=\frac{3-x}{4(x-1)^2}+\frac{x^2-2x-3}{4(x-1)^2(x+1)}$$

$$=\frac{3x+3-x^2-x+x^2-2x-3}{4(x-1)^2(x+1)}=0.$$

37. $$\frac{1}{abx}+\frac{1}{a(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$

$$=\frac{(a-b)(x-a)+bx}{abx(a-b)(x-a)}-\frac{1}{b(a-b)(x-b)}$$

$$=\frac{(ax+ab-a^2)(x-b)-ax(x-a)}{abx(a-b)(x-a)(x-b)}$$

$$=\frac{ax^2-abx+abx-ab^2-a^2x+a^2b-ax^2+a^2x}{abx(a-b)(x-c)(x-b)}=\frac{1}{x(x-a)(x-b)}.$$

38. $$\frac{s}{a}-\frac{a}{a}+\frac{s}{b}-\frac{b}{b}+\frac{s}{c}-\frac{c}{c}+\ldots\text{to } n \text{ terms}$$

$$=\frac{s}{a}+\frac{s}{b}+\frac{s}{c}+\ldots\text{ to } n\text{ terms}-(1+1+1\ldots\text{ to } n \text{ terms})$$

$$=s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots\right)-n.$$

39. $$\frac{x^4+x^2y^2-x^2y^2+y^4}{x^4-y^4}\times\frac{(x^2-y^2)^2}{x^4-2x^2y^2+y^4+x^4+2x^2y^2+y^4}$$

$$=\frac{(x^4+y^4)(x^2-y^2)(x^2-y^2)}{(x^2+y^2)(x^2-y^2)\,2\,(x^4+y^4)}=\frac{x^2-y^2}{2(x^2+y^2)}.$$

40. $\dfrac{a+x+a-x}{a+x-a+x} \div \dfrac{a^2+x^2+a^2-x^2}{a^2+x^2-a^2+x^2} = \dfrac{2a}{2x} \times \dfrac{2x^2}{2a^2} = \dfrac{x}{a}.$

41. Since $x^3+\dfrac{1}{x^3}=\left(x+\dfrac{1}{x}\right)\left(x^2-1+\dfrac{1}{x^2}\right)$; and $\dfrac{1}{x^2}-x^2=\left(\dfrac{1}{x}+x\right)\left(\dfrac{1}{x}-x\right)$

the quotient is $x^2-1+\dfrac{1}{x^2}-3\left(\dfrac{1}{x}-x\right)+4$;

or, $x^2+3x+3-\dfrac{3}{x}+\dfrac{1}{x^2}$.

The quotient may also be obtained by long division as in p. 133 of the *Algebra.*

42. $\left(\dfrac{s}{s}-\dfrac{a}{s}\right)+\left(\dfrac{s}{s}-\dfrac{b}{s}\right)+\left(\dfrac{s}{s}-\dfrac{c}{s}\right)+\ldots$ to n terms

$=(1+1+1\ldots$ to n terms$)-\dfrac{1}{s}(a+b+c+\ldots$ to n terms$)$

$=n-\dfrac{s}{s}=n-1.$

43. $\dfrac{x^2+xy-xy+y^2}{x^2-y^2} \div \dfrac{x^4-x^2y^2+x^2y^2+y^4}{(x^2+y^2)(x^2-y^2)} = \dfrac{x^2+y^2}{x^2-y^2} \times \dfrac{(x^2+y^2)(x^2-y^2)}{x^4+y^4}$

$=\dfrac{(x^2+y^2)^2}{x^4+y^4}.$

44. $\dfrac{\dfrac{x^2+2xy+y^2-2xy}{(x+y)^2}}{\dfrac{x^2-2xy+y^2+2xy}{(x-y)^2}} \div \dfrac{(x-y)^2}{(x+y)^2} = \dfrac{(x^2+y^2)(x-y)^2}{(x+y)^2(x^2+y^2)} \times \dfrac{(x+y)^2}{(x-y)^2} = 1.$

45. $(a+b)(cd-1)=(1-ab)(c+d)$, or $acd-a+bcd-b=c-abc+d-abd$,

or $a+b+c+d=bcd+acd+abd+abc$

$=abcd\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)$

$\therefore \dfrac{a+b+c+d}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}}=abcd.$

46. $\dfrac{(p+q)^4}{(p-q)^4} \div \dfrac{(p+q)^3}{(p-q)^3} = \dfrac{p+q}{p-q}.$

47. $$\frac{1-2x}{3(x^2-x+1)}+\frac{1}{6(x+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{(1-2x)(2x+2)+(x^2-x+1)}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{2x+2-4x^2-4x+x^2-x+1}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{3-3x-3x^2}{6(x^3+1)}+\frac{x+1}{2(x^2+1)}=\frac{1-x-x^2}{2(x^3+1)}+\frac{x+1}{2(x^2+1)}$$

$$=\frac{x^2-x^3-x^4+1-x-x^2+x^4+x^3+x+1}{2(x^3+1)(x^2+1)}=\frac{1}{(x^3+1)(x^2+1)}.$$

48. $$\frac{1}{x+\dfrac{z}{yz+1}}\div\frac{y}{xy+1}-\frac{1}{y(xyz+x+z)}$$

$$=\frac{yz+1}{xyz+x+z}\times\frac{xy+1}{y}-\frac{1}{y(xyz+x+z)}=\frac{xy^2z+xy+yz+1-1}{y(xyz+x+z)}=1.$$

49. $$\frac{1}{a-x}+\frac{x}{(a-x)^2}-\left(\frac{1}{a-y}+\frac{y}{(a-y)^2}\right)=\frac{a}{(a-x)^2}-\frac{a}{(a-y)^2}$$

And $$\frac{1}{(a-y)(a-x)^2}-\frac{1}{(a-y)^2(a-x)}=\frac{(a-y)-(a-x)}{(a-x)^2(a-y)^2}$$

Hence original fraction $$=\frac{a(a-y)^2-a(a-x)^2}{(a-y)-(a-x)}=a(a-y)+a(a-x)$$

$$=2a^2-ax-ay.$$

50. $$\frac{3}{abc}\div\frac{a+b-c}{abc}-\frac{3-a-b-c}{a+b-c}=\frac{3}{a+b-c}-\frac{3-a-b-c}{a+b-c}=\frac{a+b+c}{a+b-c}.$$

51. $$\frac{a+\dfrac{b^2}{b+a}}{a-\dfrac{b^2}{b-a}}(a^6-b^6)=\frac{(ab+a^2+b^2)(b-a)}{(ab-a^2-b^2)(b+a)}\cdot(a^6-b^6)$$

$$=\frac{b^3-a^3}{-(a^3+b^3)}\cdot(a^6-b^6)=\frac{a^3-b^3}{a^3+b^3}\cdot(a^3+b^3)(a^3-b^3)=(a^3-b^3)^2.$$

LXX.

1. $\left.\begin{aligned}6x+21y&=123\\6x+8y&=84\end{aligned}\right\}$ subtracting, $13y=39$, $y=3$, etc.

2. $\left.\begin{aligned}45x+72y&=909\\45x+10y&=475\end{aligned}\right\}$ subtracting, $62y=434$, $y=7$, etc.

3. $\left.\begin{aligned}26x+34y&=378\\26x+13y&=273\end{aligned}\right\}$ subtracting, $21y=105$, $y=5$, etc.

4. $\left.\begin{aligned}14x+9y&=156\\14x+4y&=116\end{aligned}\right\}$ subtracting, $5y=40$, $y=8$, etc.

5. $\left.\begin{aligned}3x+45y&=147\\3x+7y&=71\end{aligned}\right\}$ subtracting, $38y=76$, $y=2$, etc.

6. $\left.\begin{aligned}105x+133y&=924\\105x+51y&=678\end{aligned}\right\}$ subtracting, $82y=246$, $y=3$, etc.

7. $\left.\begin{aligned}6x+4y&=236\\6x+30y&=1146\end{aligned}\right\}$ subtracting, $-26y=-910$, $y=35$, etc.

8. $\left.\begin{aligned}156x+108y&=420\\156x+87y&=399\end{aligned}\right\}$ subtracting, $21y=21$, $y=1$, etc.

9. $\left.\begin{aligned}504x+98y&=2310\\504x+56y&=2184\end{aligned}\right\}$ subtracting, $42y=126$, $y=3$, etc.

LXXI.

1. $\left.\begin{aligned}6x+21y&=156\\6x-10y&=32\end{aligned}\right\}$ subtracting, $31y=124$, $y=4$, etc.

2. $\left.\begin{aligned}105x-60y&=825\\105x-91y&=763\end{aligned}\right\}$ subtracting, $31y=62$, $y=2$, etc.

3. $\left.\begin{aligned}x+y&=96\\x-y&=2\end{aligned}\right\}$ subtracting, $2y=94$, $y=47$, etc.

4. $\left.\begin{array}{l}28x+63y=553\\28x-68y=160\end{array}\right\}$ subtracting, $131y=393$, $y=3$, etc.

5. $\left.\begin{array}{l}7x+133y=679\\7x-53y=121\end{array}\right\}$ subtracting, $186y=558$, $y=3$, etc.

6. $\left.\begin{array}{l}87x-42y=525\\87x-56y=497\end{array}\right\}$ subtracting, $14y=28$, $y=2$, etc.

7. $\left.\begin{array}{l}342x-426y=1284\\342x-978y=732\end{array}\right\}$ subtracting, $552y=552$, $y=1$, etc.

8. $\left.\begin{array}{l}516x+24y=3192\\516x-731y=172\end{array}\right\}$ subtracting, $755y=3020$, $y=4$, etc.

9. $\left.\begin{array}{l}65x+117y=2444\\65x-10y=285\end{array}\right\}$ subtracting, $127y=2159$, $y=17$, etc.

LXXII.

1. $\left.\begin{array}{l}12x-21y=66\\12x-28y=-4\end{array}\right\}$ subtracting, $7y=70$, $y=10$, etc.

2. $\left.\begin{array}{l}9x-5y=52\\9x-24y=-24\end{array}\right\}$ subtracting, $19y=76$, $y=4$, etc.

3. $\left.\begin{array}{l}51x+9y=171\\51x-272y=-391\end{array}\right\}$ subtracting, $281y=562$, $y=2$, etc.

4. $\left.\begin{array}{l}21x+49y=546\\21x-57y=-408\end{array}\right\}$ subtracting, $106y=954$, $y=9$, etc.

5. $\left.\begin{array}{l}35x-21y=28\\35x-60y=-50\end{array}\right\}$ subtracting, $39y=78$, $y=2$, etc.

6. $\left.\begin{array}{l}6x+4y=78\\6x-9y=-39\end{array}\right\}$ subtracting, $13y=117$, $y=9$, etc.

7. $\left.\begin{array}{r}26x-65y=-273\\ 26x-8y=240\end{array}\right\}$ subtracting, $-57y=-513$, $y=9$, etc.

8. $\left.\begin{array}{r}105x-135y=-195\\ 105x-49y=63\end{array}\right\}$ subtracting, $-86y=-258$, $y=3$, etc.

9. $\left.\begin{array}{r}21y+36x=528\\ 21y-133x=21\end{array}\right\}$ subtracting, $169x=507$, $x=3$, etc.

LXXIII.

1. $\left.\begin{array}{r}6x+9y=24\\ 6x+14y=14\end{array}\right\}$ subtracting, $-5y=10$, $y=-2$, etc.

2. $\left.\begin{array}{r}15x-6y=153\\ 38x-6y=360\end{array}\right\}$ subtracting, $-23x=-207$, $x=9$, etc.

3. $\left.\begin{array}{r}6x-10y=102\\ 6x+21y=9\end{array}\right\}$ subtracting, $-31y=93$, $y=-3$.

4. $\left.\begin{array}{r}35y-15x=695\\ 35y+14x=637\end{array}\right\}$ subtracting, $-29x=58$, $x=-2$, etc.

5. $\left.\begin{array}{r}8x+18y=212\\ 8x+17y=198\end{array}\right\}$ subtracting, $y=14$, etc.

6. $\left.\begin{array}{r}18x-63y=72\\ 18x-8y=-38\end{array}\right\}$ subtracting, $-55y=110$, $y=-2$, etc.

7. $\left.\begin{array}{r}17x+12y=59\\ 57x-12y=459\end{array}\right\}$ adding, $74x=518$, $x=7$, etc.

8. $\left.\begin{array}{r}24x+9y=9\\ 12x+9y=3\end{array}\right\}$ subtracting, $12x=6$, $x=\frac{1}{2}$, etc.

9. $\left.\begin{array}{r}238x-966y=-1442\\ 238x-221y=-697\end{array}\right\}$ subtracting, $-745y=-745$, $y=1$, etc.

LXXIV.

1. $\left.\begin{array}{l}3x+2y=42\\2x+3y=48\end{array}\right\}$, $\left.\begin{array}{l}6x+4y=84\\6x+9y=144\end{array}\right\}$, $-5y=-60$, $y=12$, etc.

2. $\left.\begin{array}{l}30x+y=630\\20y-x=580\end{array}\right\}$, $\left.\begin{array}{l}600x+20y=12600\\20y-x=580\end{array}\right\}$, $601x=12020$, $x=20$, etc.

3. $\left.\begin{array}{l}x+49y=1757\\y+49x=2093\end{array}\right\}$, $\left.\begin{array}{l}49x+2401y=86093\\49x+y=2093\end{array}\right\}$, $2400y=84000$, $y=35$, etc.

4. $\left.\begin{array}{l}x+y+15=30\\x-y+14=19\end{array}\right\}$, $\left.\begin{array}{l}x+y=15\\x-y=5\end{array}\right\}$, $2x=20$, $x=10$, etc.

5. $\left.\begin{array}{l}14x+5y=826\\39x-14y=-1609\end{array}\right\}$, $\left.\begin{array}{l}196x+70y=11564\\195x-70y=-8045\end{array}\right\}$,

$391x=3519$, $x=9$, etc.

6. $\left.\begin{array}{l}6x+9y=150-5y\\8y-6x=9x+12\end{array}\right\}$, $\left.\begin{array}{l}6x+14y=150\\8y-15x=12\end{array}\right\}$, $\left.\begin{array}{l}30x+70y=750\\16y-30x=24\end{array}\right\}$,

$86y=774$, etc.

7. $\left.\begin{array}{l}7x-y+2=35\\12y-x-10=9\end{array}\right\}$, $\left.\begin{array}{l}7x-y=33\\12y-x=19\end{array}\right\}$, $\left.\begin{array}{l}7x-y=33\\84y-7x=133\end{array}\right\}$, $83y=166$, etc.

8. $\left.\begin{array}{l}x+32=2y-48\\12x+12y+20y=30x-15y+2100\end{array}\right\}$, $\left.\begin{array}{l}x-2y=-80\\47y-18x=2100\end{array}\right\}$,

$\left.\begin{array}{l}18x-36y=-1440\\47y-18x=2100\end{array}\right\}$, adding, $11y=660$, $y=60$, etc.

9. $\left.\begin{array}{l}15x-25y+30=4x+2y\\96-3x+6y=6x+4y\end{array}\right\}$, $\left.\begin{array}{l}11x-27y=-30\\2y-9x=-96\end{array}\right\}$,

$\left.\begin{array}{l}99x-243y=-270\\22y-99x=-1056\end{array}\right\}$, adding, $-221y=-1326$, $y=6$, etc.

10. $\left.\begin{array}{r}x+2+24y=93\\ y+5+40x=768\end{array}\right\}$, $\left.\begin{array}{r}x+24y=91\\ 40x+y=763\end{array}\right\}$, $\left.\begin{array}{r}40x+960y=3640\\ 40x+y\quad=763\end{array}\right\}$,

$959y=2877$, etc.

11. $\left.\begin{array}{l}2x-y+21x=14y-42\\ 6y+18+5y-5x=60x-240\end{array}\right\}$, $\left.\begin{array}{r}23x-15y=-\ 42\\ 11y-65x=-258\end{array}\right\}$,

$\left.\begin{array}{r}253x-165y=-\ 462\\ 165y-975x=-3870\end{array}\right\}$ adding, $-722x=-4332$, $x=6$, etc.

12. $\left.\begin{array}{l}12x-24-200+20x=15y-150\\ 16y+32=12x+3y+39\end{array}\right\}$, $\left.\begin{array}{r}32x-15y=74\\ 13y-12x=7\end{array}\right\}$,

$\left.\begin{array}{r}96x-45y=222\\ 104y-96x=56\end{array}\right\}$, adding, $59y=278$, $y=\dfrac{278}{59}$, etc.

13. $\left.\begin{array}{l}5x-6y+39x=52y-26\\ 10x+12y-9x+6y=24y-24\end{array}\right\}$, $\left.\begin{array}{r}44x-58y=-26\\ x-\ 6y=-24\end{array}\right\}$

$\left.\begin{array}{r}22x-29y=-\ 13\\ 22x-132y=-528\end{array}\right\}$, $103y=515$, $y=5$, etc.

14. $\left.\begin{array}{l}15x-9-9x+57=24-6y+2x\\ 16x+8y-18x+14=12y+36-4x-5y\end{array}\right\}$, $\left.\begin{array}{r}4x+6y=-24\\ 2x+\ y=22\end{array}\right\}$,

$\left.\begin{array}{r}2x+3y=-12\\ 2x+\ y=22\end{array}\right\}$, $2y=-34$, $y=-17$, etc.

15. $\left.\begin{array}{l}4x+5y=40x-40y\\ 4x-2y+12y=3\end{array}\right\}$, $\left.\begin{array}{r}45y-36x=0\\ 4x+10y=3\end{array}\right\}$, $\left.\begin{array}{r}5y-\ 4x=0\\ 4x+10y=3\end{array}\right\}$, $15y=3$, etc.

LXXV.

1. $\left.\begin{array}{l}mpx+npy=ep\\ mpx+mqy=fm\end{array}\right\}$ subtracting $(np-mq)y=ep-fm$, etc.

2. $\left.\begin{array}{l}adx+bdy=cd\\ adx-aey=af\end{array}\right\}$ subtracting $(bd+ae)y=cd-af$, etc.

3. $\left.\begin{array}{l} acx - bcy = cm \\ acx + aey = an \end{array}\right\}$, $y(ae + bc) = an - cm$, etc.

4. $\left.\begin{array}{l} cx - dy = 0 \\ cx + cy = ce \end{array}\right\}$ $(c + d)\,y = ce$, etc.

5. $\left.\begin{array}{l} mm'x - m'ny = m'r \\ mm'x + mn'y = mr' \end{array}\right\}$, $(mn' + m'n)\,y = mr' - m'r$, etc.

6. $\left.\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$, $2x = e + b$, $x = \dfrac{a + b}{2}$, etc.

7. $\left.\begin{array}{l} adx + bdy = cd \\ adx + afy = ac^2 \end{array}\right\}$, $(af - bd)\,y = ac^2 - cd$, etc.

8. $\left.\begin{array}{l} abdx + cd^2y = 2d \\ abdx - bcdy = d - b \end{array}\right\}$, $cdy\,(d + b) = d + b$, etc.

9. $\left.\begin{array}{l} 3a^2 + ax = b^2 + by \\ ax + 2by = d \end{array}\right\}$, $\left.\begin{array}{l} ax - \ \ by = b^2 - 3a^2 \\ ax + 2by = d \end{array}\right\}$, $3by = 3a^2 - b^2 + d$, etc.

10. $\left.\begin{array}{l} bcx - cy = -2b \\ b^3cy + ac^3 - ab^3 = 2b^4 + bc^4x \end{array}\right\}$, $\left.\begin{array}{l} bc^4x - c^4y \ = -2bc^3 \\ b^3cy - bc^4x = 2b^4 - ac^3 + ab^3 \end{array}\right\}$,

$$b^3cy - c^4y = 2b^4 - ac^3 + ab^3 - 2bc^3,$$

$$y = \frac{2b^4 - ac^3 + ab^3 - 2bc^3}{c\,(b^3 - c^3)} = \frac{2b + a}{c}, \text{ etc.}$$

11. $\left.\begin{array}{l} bx + bc - b^2 + cx + c^2 - bc + ay + a^2 = 2a^2 \\ a^3y = (b + c)^2\,(b - c)\,x \end{array}\right\}$,

$\left.\begin{array}{l} (b + c)\,x + ay = a^2 + b^2 - c^2 \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\}$, $\left.\begin{array}{l} a^2(b + c)x + a^3y = a^2(a^2 + b^2 - c^2) \\ (b + c)\,(b^2 - c^2)\,x - a^3y = 0 \end{array}\right\}$,

$$(b + c)\,(a^2 + b^2 - c^2)\,x = a^2\,(a^2 + b^2 - c^2)\ ;\ x = \frac{a^2}{b + c}, \text{ etc.}$$

12. $\left.\begin{array}{l} 3x + 5y = \dfrac{8b^2m - 2bm^2}{b^2 - m^2} \\ (b^2 - m^2)x + (b + c + m)\,my = b^2m + 2bm^2 + \dfrac{bcm^2}{b + m} \end{array}\right\}$

$$\left.\begin{array}{l}3\,(b^2-m^2)\,x+5\,(b^2-m^2)\,y=8b^2m-2bm^2\\ 3(b^2-m^2)x+3(bm+cm+m^2)=3b^2m+6bm^2+\dfrac{3bcm^2}{b+m}\end{array}\right\}$$

$$(5b^2-3bm-3cm-8m^2)\,y=5b^2m-8bm^2-\frac{3bcm^2}{b+m}$$

$$=\frac{5b^3m+5b^2m^2-8b^2m^2-8bm^3-3bcm^2}{b+m}$$

$$=\frac{bm\,(5b^2-3bm-3cm-8m^2)}{b+m}\text{, etc.}$$

LXXVI.

1. $$\left.\begin{array}{l}\dfrac{4}{x}+\dfrac{8}{y}=40\\ \dfrac{4}{x}+\dfrac{3}{y}=20\end{array}\right\}\text{ subtracting, }\frac{5}{y}=20\,;\ y=\frac{1}{4}\text{, etc.}$$

2. $$\left.\begin{array}{l}\dfrac{3}{x}+\dfrac{6}{y}=3a\\ \dfrac{3}{x}+\dfrac{4}{y}=b\end{array}\right\}\text{ subtracting, }\frac{2}{y}=3a-b\,;\ y=\frac{2}{3a-b}\text{, etc.}$$

3. $$\left.\begin{array}{l}\dfrac{ab}{x}+\dfrac{b^2}{y}=bc\\ \dfrac{ab}{x}+\dfrac{a^2}{y}=ad\end{array}\right\}\text{ subtracting, }\frac{b^2-a^2}{y}=bc-ad\,;\ y=\frac{b^2-a^2}{bc-ad}\text{, etc.}$$

4. $$\left.\begin{array}{l}\dfrac{a}{x}+\dfrac{b}{y}=m\\ \dfrac{a}{x}-\dfrac{b}{y}=n\end{array}\right\}\text{ subtracting, }\frac{2b}{y}=m-n\,;\ y=\frac{2b}{m-n}\text{, etc.}$$

5. $$\left.\begin{array}{l}\dfrac{56}{x}+\dfrac{40}{y}=152\\ \dfrac{56}{x}-\dfrac{21}{y}=\ \ 49\end{array}\right\}\text{ subtracting, }\frac{61}{y}=103\,;\ y=\frac{61}{103}\text{, etc.}$$

6. Multiply the second equation by 4, and we get

$$\left.\begin{array}{l}\dfrac{5}{3x}+\dfrac{2}{5y}=\ 7\\[2ex] \dfrac{14}{3x}-\dfrac{2}{5y}=12\end{array}\right\}\text{ adding, }\frac{19}{3x}=19\text{ ; }x=\frac{1}{3},\text{ etc.}$$

7. $$\left.\begin{array}{l}\dfrac{4}{ax}+\dfrac{6}{by}=10\\[2ex] \dfrac{15}{ax}-\dfrac{6}{by}=\ 9\end{array}\right\}\text{ adding, }\frac{19}{ax}=19\text{ ; }x=\frac{1}{a},\text{ etc.}$$

8. Multiply the first equation by $\frac{n^2}{m}$, and we get

$$\left.\begin{array}{l}\dfrac{n}{x}+\dfrac{n^3}{m^2y}=\dfrac{mn^2+n^3}{m}\\[2ex] \dfrac{n}{x}+\dfrac{m}{y}\ =m^2+n^2\end{array}\right\}\text{ subtracting, }\frac{n^3-m^3}{m^2y}=\frac{n^3-m^3}{m}\text{ ; }y=\frac{1}{m},\text{ etc.}$$

LXXVII.

1. From the first and second equations we get

$$\left.\begin{array}{r}5x+7y-2z=13\\ 16x+6y+2z=34\end{array}\right\}\text{ adding, }21x+13y=47\ (1)$$

From the first and third equations we get

$$\left.\begin{array}{r}25x+35y-10z=65\\ x-\ 4y+10z=23\end{array}\right\}\text{ adding, }26x+31y=88\ (2)$$

Then from (1) and (2) we find $x=1$, $y=2$; and hence $z=3$.

2. From the first and second equations we get

$$\left.\begin{array}{r}5x+3y-6z=\ 4\\ 9x-3y+6z=24\end{array}\right\}\text{ adding, }14x=28\text{ ; }x=2$$

From the second and third we get $2x+y=6$; $\therefore y=2$, etc.

3. From the first and second equations we get

$$\left.\begin{array}{r}15x-9y+6z=63\\ 16x-2y-6z=\ 6\end{array}\right\}\text{ adding, }31x-11y=69\ (1).$$

From the first and third, subtracting, $3x-6y=-18$ (2)

Then from (1) and (2) we find $x=4$, $y=5$; and $\therefore z=8$.

4. From the first and second equations we get

$$\left.\begin{aligned}4x-5y+2z&=6\\4x+6y-2z&=40\end{aligned}\right\}\text{ adding, }8x+y=46\ (1)$$

From the second and third equations we get

$$\left.\begin{aligned}6x+9y-3z&=60\\7x-4y+3z&=35\end{aligned}\right\}\text{ adding, }13x+5y=95\ (2)$$

Then from (1) and (2) we find $x=5$, $y=6$; and $\therefore z=8$.

5. From the second and first equations we get

$$\left.\begin{aligned}5x+4y+3z&=22\\3x+3y+3z&=18\end{aligned}\right\}\text{ subtracting, }2x+y=4\ (1)$$

From the third and second equations we get

$$\left.\begin{aligned}15x+10y+6z&=53\\10x+8y+6z&=44\end{aligned}\right\}\text{ subtracting, }5x+2y=9\ (2)$$

Then from (1) and (2) we find $x=1$, $y=2$; and $\therefore z=3$.

6. From the first and second equations we get

$$\left.\begin{aligned}8x+4y-3z&=6\\3x+9y-3z&=21\end{aligned}\right\}\text{ subtracting, }5x-5y=-15\ (1)$$

From the second and third equations we get

$$\left.\begin{aligned}4x+12y-4z&=28\\4x-5y+4z&=8\end{aligned}\right\}\text{ adding, }8x+7y=36\ (2)$$

Then from (1) and (2) we find $x=1$, $y=4$; and $\therefore z=6$.

7. From the second and first equations we get

$$\left.\begin{aligned}8x+4y+2z&=50\\2x+2y+2z&=60\end{aligned}\right\}\text{ subtracting, }6x+2y=-10\ (1)$$

From the third and first equations we get

$$\left.\begin{aligned} 27x+9y+3z&=64 \\ 3x+3y+3z&=90 \end{aligned}\right\}\text{ subtracting, } 24x+6y=-26\ (2)$$

Then from (1) and (2) we find $x=\frac{2}{3}$, $y=-7$; and $\therefore z=36\frac{1}{3}$.

8. From the first and second equations we get

$$\left.\begin{aligned} 4x-3y+z&=9 \\ 27x+3y-15z&=48 \end{aligned}\right\}\text{ adding, } 31x-14z=57\ (1)$$

From the second and third equations we get

$$\left.\begin{aligned} 36x+4y-20z&=64 \\ x-4y+3z&=2 \end{aligned}\right\}\text{ adding, } 37x-17z=66\ (2)$$

Then from (1) and (2) we find $x=5$, $z=7$; and $\therefore y=6$.

9. From the first and second equations we get

$$\left.\begin{aligned} 24x+10y-8z&=58 \\ 65x-10y+25z&=290 \end{aligned}\right\}\text{ adding, } 89x+17z=348\ (1)$$

From the first and third equations we get

$$\left.\begin{aligned} 12x+5y-4z&=29 \\ 85x-5y-5z&=75 \end{aligned}\right\}\text{ adding, } 97x-9z=104\ (2)$$

Then from (1) and (2) we find $x=2$, $z=10$; and $\therefore y=9$.

10. Subtracting the second equation from the first, we get $2y=20$; $y=10$.

Adding the third equation to the second, we get $2z=10$; $z=5$.

LXXVIII.

1. Let x and y be the numbers; then $x+y=28$; $x-y=4$, etc.

2. Let x and y be the numbers; then $x+y=256$; $x-y=10$, etc.

3. Let x and y be the numbers; then $x+y=13{\cdot}5$; $x-y=1$, etc.

4. Let x and y be the numbers; then $7x+5y=332$;
$51x-51y=408$, etc.

5. Let x be the age of the father, and y the age of the son.
Then $x-7=4(y-7)$ and $x+7=2(y+7)$, etc.

6. Let x, y, z be the numbers; then $x+y=70, x+z=80, y+z=90$, etc.

7. Let x and y be the sums contributed by A and B;
then $400-(x+y)$ is the sum contributed by C.
Then $y=2x+20$, and $400-(x+y)=x+y$, etc.

8. Let x be A's money, and y B's money in shillings.
Then $3(x-10)=y+10$, and $x+10=2(y-10)$, etc.

9. Let x and y be A's and B's shares; then $760-(x+y)$ is C's share.
Then $x+y=760-x-y+240$; $y+760-x-y=x+360$, etc.

10. Let x and y be the numbers; then $\frac{x+y}{2}=24$; $\frac{x-y}{2}=17$, etc.

11. Let x be the greater number; and y the less; then
$\frac{x}{y}=4+\frac{3}{y}$; $\frac{x+y+38}{x}=2+\frac{2}{x}$, etc.

12. Let x be the first part, y the second, and $\therefore$ $144-(x+y)$ the third.
Then $\frac{x}{y}=3+\frac{2}{y}$; $\frac{144-x-y}{x+y}=2+\frac{6}{x+y}$, etc.

13. Let x be A's money and y B's money in pounds.
Then $x+\frac{y}{2}=120$; $\frac{2x}{3}+y=120$, etc.

14. Let x be the age of the father, y the age of the son.
Then $x-12=y+12$; $x+12=3(y-12)$, etc.

15. Let x be the greater number, and y the less.
Then $3x = 2x + 10$, and $2x + 3y = 24$, etc.

16. Let x be the age of the father, and y the age of the son.
Then $x + y = \frac{1}{2}(x + y + 50)$; $x - y = \frac{1}{3}(x + y + 40)$, etc.

17. Let x be the greater number, and y the less.
Then $\frac{y}{x} = \cdot 21 + \frac{\cdot 0057}{x}$; $\frac{x}{y} = 4 + \frac{\cdot 742}{y}$, etc.

18. Let x be the cost of a barrel of beer, y of a barrel of porter in pounds.
Then $6x + 10y = 51$; $3x + 7y = 32\frac{1}{10}$. Hence we find $x = 3$, and hence 10 barrels of beer can be bought for £30.

19. Let x be the cost of 1 lb. of tea, y of 1 lb. of coffee in pence.
Then $7x + 5y = 352$, and $4x + 9y = 324$, etc.

20. Let x be the cost of a horse, y of a cow in pounds.
Then $12x + 14y = 380$, and $5x + 3y = 130$, etc.

21. Let x be the cost of a yard of silk, y of a yard of cloth in pence.
Then $8x + 19y = 4370$, and $20x + 16y = 6200$, etc.

22. In one day 10 men and 6 women earn £3, 3s.
In one day 4 men and 8 women earn £2, 2s.
Let x be the daily earnings of a man, y of a woman in shillings.
Then $10x + 6y = 63$, and $4x + 8y = 42$, etc.

23. Let x be the number at £37; and y the number at £45.
Then $37x + 45y = 4220$; and $x + y = 100$, etc.

24. Let $10x + y$ be the number.
Then $x + y = 8$; and $10x + y + 36 = 10y + x$, etc.

25. Let $10x + y$ be the number.
Then $x + y = 10$; and $10x + y + 54 = 10y + x$, etc.

26. Let $10x + y$ be the number.
Then $x + y = 9$; and $10x + y + 9 = 10y + x$, etc.

27. Let $10x+y$ be the number.

Then $x+y=6$; and $\frac{10x+y}{x+y}=4$, etc.

28. Let $10x+y$ be the number.

Then $x+y=9$; and $\frac{10x+y}{x+y}=5$, etc.

29. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y}=7$; and $\frac{10y+x-12}{x-y}=9$, etc.

30. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y}=6+\frac{3}{x+y}$; and $\frac{10y+x}{x+y}=4+\frac{9}{x+y}$

or, $10x+y=6x+6y+3$; and $10y+x=4x+4y+9$, etc.

31. Let $10x+y$ be the number.

Then $\frac{10x+y}{x+y-2}=5+\frac{1}{x+y-2}$; $\frac{10y+x}{x+y+2}=5+\frac{8}{x+y+2}$, etc.

32. Let $10x+y$ and $10y+x$ be the numbers.

Then $10x+y+9=10y+x$; and $10x+y+10y+x=33$, etc.

33. Let $100x+10x+x$ be the number.

Then $100x+10x+x=37x^2$; $\therefore x=3$, etc.

34. Let $100x+10y+z$ be the number.

Then $y=2z$; $x+z=9$; $x+y+z=17$.

Hence $x=9-z$; and $\therefore 9-z+2z+z=17$; $z=4$, etc.

35. Let $100x+10y+z$ be the number.

Then $x+y+2=21$; $x+y=z+3$;

$100x+10y+z+198=100z+10y+x$, etc.

36. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+7}{y}=2$, and $\frac{x}{y-1}=1$, etc.

37. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, etc.

38. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+1}=\frac{1}{3}$, etc.

39. Let $\frac{x}{y}$ be the fraction. Then $x+1=y$, and $x=\frac{1}{2}(y+1)$, etc.

40. Let $\frac{x}{y}$ be the fraction. Then $\frac{x-3}{y-3}=\frac{1}{4}$, and $\frac{x+5}{y+5}=\frac{1}{2}$, etc.

41. Let $\frac{x}{y}$ be the fraction. Then $\frac{x}{y+4}=\frac{7}{9}$, and $\frac{x-15}{y}=\frac{20}{41}$, etc.

42. Let $\frac{x}{y}$ be the fraction. Then $\frac{x+1}{y}=\frac{1}{2}$, and $\frac{x}{y+17}=\frac{1}{3}$, etc.

43. Let x and y be the sums invested.

Then $\frac{x\times 5}{100}=$ income on first investment.

and $\frac{x\times 4}{100}=$ income on second investment.

$\therefore \frac{x\times 5}{100}=\frac{x\times 4}{100}+10$, etc.

44. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 10}{100\times 12}=5250$

and $x+\frac{x\times y\times 18}{100\times 12}=5450.$

$$\text{Hence }\left.\begin{aligned}120x+xy&=630000\\ \text{and }\ 200x+3xy&=1090000\end{aligned}\right\},\ \left.\begin{aligned}360x+3xy&=1890000\\ 200x+3xy&=1090000\end{aligned}\right\},$$

$$160x=800000, \text{ etc.}$$

45. Let x be the sum invested, and y the rate of interest.

Then $x+\frac{x\times y\times 6}{100}=5200.$

and $x+\frac{x\times y\times 10}{100}=6000$

$$\text{Hence }\left.\begin{aligned}50x+3xy&=260000\\ 10x+xy&=60000\end{aligned}\right\},\ \left.\begin{aligned}50x+3xy&=260000\\ 30x+3xy&=180000\end{aligned}\right\},$$

$$20x=80000, \text{ etc.}$$

46. Let x be the number of quarts of the first, y of the second.

Then $\left.\begin{array}{r} x+\ y\ =50 \\ 36x+20y=30\times 50 \end{array}\right\}, \left.\begin{array}{r} 36x+36y=1800 \\ 36x+20y=1500 \end{array}\right\}, 16y=300$, etc.

47. Let x be the number of lbs. of the cheaper, y of the dearer.

Then $x+y=30$

and $20(x+y)=14x+18y+10x$, etc.

48. Let x be the rate of the rowing in miles an hour in still water, y the rate of the stream in miles an hour.

Then $x+y=12$, $x-y=6$, etc.

49. Let x be the distance in miles, y the rate in miles an hour of pulling.

Then $x=1\frac{2}{3}(y+4)$

and $x-3=4\frac{1}{4}(y-4)$, etc.

50. Let x be the number of leaps the hare takes, and y the length of each in feet.

Then $\frac{5x}{6}$ is the number of leaps the dog takes, and $\frac{9y}{7}$ the length of each in feet.

Then $50y+xy=\frac{5x}{6}\times\frac{9y}{7}$

Divide by y; then $50+x=\frac{15x}{14}$; $x=700$.

51. Let x be the number of leaps the dog takes, and y the length of each in feet.

Then $\frac{4x}{3}$ is the number of leaps the hare takes, and $\frac{2y}{3}$ is the length of each in feet.

Then $50y+\frac{4x}{3}\times\frac{2y}{3}=xy$.

Divide by y; then $50+\frac{8x}{9}=x$; $x=450$, etc.

52. Let x be the number of apples, and y the number of pears.

Then $\frac{x}{4}+\frac{y}{5}=30$; and $\frac{x}{8}+\frac{y}{15}=13$, etc.

53. Let x be the number of men, y the reckoning of each in shillings.

$$\begin{array}{l}\text{Then} & (x+3)(y-1)=xy \\ \text{and} & (x-2)(y+1)=xy\end{array}\Bigg\}, \quad \left.\begin{array}{l}xy+3y-x-3=xy \\ xy-2y+x-2=xy\end{array}\right\},$$

$$\left.\begin{array}{r}3y-x=3 \\ x-2y=2\end{array}\right\}\text{ adding, } y=5, \text{ and hence } x=12.$$

54. Let x be the number that voted for A and C,

y „ „ for A and B,

z „ „ for A only.

$$\text{Then } \left.\begin{array}{r}x+y+z=1056 \\ x+85+98=933 \\ y+85+744=987\end{array}\right\}, \text{ hence } x=750,\ y=158,\ z=148.$$

55. Let x be the distance in miles; y the rate in miles an hour.

$$\left.\begin{array}{ll}\text{Then} & \frac{x}{y+\frac{1}{2}}=\frac{x}{y}-1\frac{1}{2} \\ \text{and} & \frac{x}{y-\frac{1}{2}}=\frac{x}{y}+2\frac{1}{2}\end{array}\right\}, \text{ or, } \left.\begin{array}{l}\frac{2x}{2y+1}=\frac{2x-3y}{2y} \\ \frac{2x}{2y-1}=\frac{2x+5y}{2y}\end{array}\right\}$$

$$\left.\begin{array}{l}4xy=4xy+2x-6y^2-3y \\ 4xy=4xy-2x+10y^2-5y\end{array}\right\}, \quad \left.\begin{array}{l}2x=6y^2+3y \\ 2x=10y^2-5y\end{array}\right\}$$

hence $6y^2+3y=10y^2-5y$; $8y=4y^2$; $8=4y$; $y=2$, etc.

56. Suppose the first crew pulls x strokes of y yards each in a minute.

Then the second crew pulls $\frac{8x}{9}$ strokes of $\frac{90y}{79}$ yards each in a minute.

Hence in one minute the first crew pulls over xy yards,

and in one minute the second crew pulls over $\frac{80xy}{79}$ yards;

∴ the second is the faster crew.

Again, the second crew gains $\frac{xy}{79}$ yards in a minute;

∴ it gains $\frac{xy}{79}\div\frac{8x}{9}$ yards in a stroke, or, $\frac{9y}{79\times 8}$ yards.

Now it has to gain $\frac{4\times 90y}{79}$ yards;

$\therefore$ it must take $\frac{4\times 90y}{79} \div \frac{9y}{79\times 8}$ strokes, or, 320 strokes.

57. Let x be the rate of the sculler; y the rate of the barges.

Then $\frac{b}{x}$ = time he takes to meet the first barge

$\frac{a-b}{y}$ = time the first barge takes to meet him.

Hence $\frac{b}{x}=\frac{a-b}{y}$, or, $\frac{b}{a-b}=\frac{x}{y}$. (1)

Also $\frac{b'}{x}$ = time he takes to overtake the second barge

$\frac{b'-a}{y}$ = time the second barge is in motion before he overtakes it;

$\therefore \frac{b'}{x}=\frac{b'-a}{y}$, or, $\frac{b'}{b'-a}=\frac{x}{y}$ (2)

From (1) and (2) $\frac{b}{a-b}=\frac{b'}{b'-a}$, or, $bb'-ab=ab'-bb'$,

or $2bb'=ab+ab'$, or, $\frac{2bb'}{abb'}=\frac{ab}{abb'}+\frac{ab'}{abb'}$, or, $\frac{2}{a}=\frac{1}{b'}+\frac{1}{b}$.

58. Let x be the number of feet passed over by the longer train in a second,

y the number of feet passed over by the shorter train in a second.

Then $\frac{3x}{2}+\frac{3y}{2}=176$, and $6x-6y=176$.

Hence $x=\frac{220}{3}$ and $y=44$.

Hence the longer train goes in miles per hour $\frac{220\times 60\times 60}{3\times 3\times 1760}$, or, 50.

and the shorter train goes in miles per hour $\frac{44\times 60\times 60}{3\times 1760}$, or, 30.

59. Let x and y be the circumferences of the wheels in yards.

$$\text{Then } \frac{120}{x} = \frac{120}{y} + 6, \text{ and } \frac{120}{x + \frac{x}{4}} = \frac{120}{y + \frac{y}{5}} + 4;$$

$$\text{hence } \frac{20}{x} - \frac{20}{y} = 1, \text{ and } \frac{96}{x} - \frac{100}{y} = 4; \therefore x = 4, y = 5.$$

60. Let x be the number of hours he takes to go.

Then $10 - x$ is the number of hours he takes to return,

and $x = \frac{2}{3}(10 - x)$; whence $x = 4$, and $10 - x = 6$.

Again let y be the number of miles per hour he can row in still water, and z the number of miles per hour the stream flows.

Then $y + z = 5$, and $y - z = \frac{20}{6}$; whence $z = \frac{5}{6}$.

61. Let the digits commencing with the left be $1, a, b, c, d, e$.

Then the number is $100000 + 10000a + 1000b + 100c + 10d + e$.

When the 1 is removed to the unit's place
the new number is $100000a + 10000b + 1000c + 100d + 10e + 1$.

Then $100000a + 10000b + 1000c + 100d + 10e + 1$
$= 300000 + 30000a + 3000b + 300c + 30d + 3e$

$\therefore$ $70000a + 7000b + 700c + 70d + 7e = 299999$
or, $10000a + 1000b + 100c + 10d + e = 42857$
$\therefore$ $100000 + 10000a + 1000b + 100c + 10d + e = 142857$
that is, the required number is 142857.

LXXX.

1.
$$\begin{array}{r|l}
 & 4a^2 + 12ab + 9b^2 \;(2a + 3b \\
 & 4a^2 \\ \hline
4a + 3b & \quad 12ab + 9b^2 \\
 & \quad 12ab + 9b^2 \\ \hline
\end{array}$$

2.
$$\begin{array}{r|l}
 & 16k^{10} - 24k^5l^3 + 9l^6 \;(4k^5 - 3l^3 \\
 & 16k^{10} \\ \hline
8k^5 - 3l^3 & \quad -24k^5l^3 + 9l^6 \\
 & \quad -24k^5l^3 + 9l^6 \\ \hline
\end{array}$$

3.

$$\begin{array}{r|l} & a^2b^2+162ab+6561\ (ab+81 \\ & a^2b^2 \\ \cline{2-2} 2ab+81 & \qquad 162ab+6561 \\ & \qquad 162ab+6561 \\ \cline{2-2} \end{array}$$

4.

$$\begin{array}{r|l} & y^6-38y^3+361\ (y^3-19 \\ & y^6 \\ \cline{2-2} 2y^3-19 & -38y^3+361 \\ & -38y^3+361 \\ \cline{2-2} \end{array}$$

5.

$$\begin{array}{r|l} & 9a^2b^2c^2-102abc+289\ (3abc-17 \\ & 9a^2b^2c^2 \\ \cline{2-2} 6abc-17 & \qquad -102abc+289 \\ & \qquad -102abc+289 \\ \cline{2-2} \end{array}$$

6.

$$\begin{array}{r|l} & x^4-6x^3+19x^2-30x+25\ (x^2-3x+5 \\ & x^4 \\ \cline{2-2} 2x^2-3x & -6x^3+19x^2 \\ & -6x^3+\ \ 9x^2 \\ \cline{2-2} 2x^2-6x+5 & \qquad\qquad 10x^2-30x+25 \\ & \qquad\qquad 10x^2-30x+25 \\ \cline{2-2} \end{array}$$

7.

$$\begin{array}{r|l} & 9x^4+12x^3+10x^2+4x+1\ (3x^2+2x+1 \\ & 9x^4 \\ \cline{2-2} 6x^2+2x & \qquad 12x^3+10x^2 \\ & \qquad 12x^3+\ \ 4x^2 \\ \cline{2-2} 6x^2+4x+1 & \qquad\qquad\qquad 6x^2+4x+1 \\ & \qquad\qquad\qquad 6x^2+4x+1 \\ \cline{2-2} \end{array}$$

8.

$$\begin{array}{r|l} & 4r^4-12r^3+13r^2-6r+1\ (2r^2-3r+1 \\ & 4r^4 \\ \cline{2-2} 4r^2-3r & -12r^3+13r^2 \\ & -12r^3+\ \ 9r^2 \\ \cline{2-2} 4r^2-6r+1 & \qquad\qquad\qquad 4r^2-6r+1 \\ & \qquad\qquad\qquad 4r^2-6r+1 \\ \cline{2-2} \end{array}$$

9.

$$\begin{array}{l|l}
 & 4n^4+4n^3-7n^2-4n+4\ (2n^2+n-2 \\
 & 4n^4 \\ \hline
4n^2+n & 4n^3-7n^2 \\
 & 4n^3+\ n^2 \\ \hline
4n^2+2n-2 & \quad -8n^2-4n+4 \\
 & \quad -8n^2-4n+4 \\ \hline
\end{array}$$

10.

$$\begin{array}{l|l}
 & 1\ -6x+13x^2+12x^3+4x4\ (1-3x+2x^2 \\
 & 1 \\ \hline
2-3x & -6x+13x^2 \\
 & -6x+\ 9x^2 \\ \hline
2-6x+2x^2 & \quad 4x^2+12x^3+4x^4 \\
 & \quad 4x^4-12x^3+4x^4 \\ \hline
\end{array}$$

11.

$$\begin{array}{l|l}
 & x^6-4x^5+10x^4-12x^3+9x^2\ (x^3-2x^2+3x \\
 & x6 \\ \hline
2x^3-2x^2 & -4x^5+10x^4 \\
 & -4x^5+\ 4x^4 \\ \hline
2x^3-4x^2+3x & \quad 6x^4-12x^3+9x^2 \\
 & \quad 6x^4-12x^3+9x^2 \\ \hline
\end{array}$$

12.

$$\begin{array}{l|l}
 & 4y^4-12y^3z+25y^2z^2-24yz^3+16z^4\ (2y^2-3yz+4z \\
 & 4y^4 \\ \hline
4y^2-3yz & -12y^3z+25y^2z^2 \\
 & -12y^3z+9y^2z^2 \\ \hline
4y^2-6yz+4z^2 & \quad 16y^2z^2-24yz^3+16z^4 \\
 & \quad 16y^2z^2-24yz^3+16z^4 \\ \hline
\end{array}$$

13. $a^2+4ab+4b^2+9c^2+6ac+12bc\,(a+2b+3c$

a^2

$2a+2b$ $\quad 4ab+4b^2$

$\quad 4ab+4b^2$

$2a+4b+3c$ $\quad 6ac+12bc+9c^2$

$\quad 6ac+12bc+9c^2$

14. $a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6\,(a^3+a^2b+ab^2+b^3$

a^6

$2a^3+a^2b$ $\quad 2a^5b+3a^4b^2$

$\quad 2a^5b+a^4b^2$

$2a^3+2a^2b+ab^2$ $\quad 2a^4b^2+4a^3b^3+3a^2b^4$

$\quad 2a^4b^2+2a^3b^3+a^2b^4$

$2a^3+2a^2b+2ab^2+b^3$ $\quad 2a^3b^3+2a^2b^4+2ab^5+b^6$

$\quad 2a^3b^3+2a^2b^4+2ab^5+b^6$

15. $x^6-4x^5+6x^3+8x^2+4x+1\,(x^3-2x^2-2x-1$

x^6

$2x^3-2x^2$ $\quad -4x^5+6x^3$

$\quad -4x^5+4x^4$

$2x^3-4x^2-2x$ $\quad -4x^4+6x^3+8x^2$

$\quad -4x^4+8x^3+4x^2$

$2x^3-4x^2-4x-1$ $\quad -2x^3+4x^2+4x+1$

$\quad -2x^3+4x^2+4x+1$

16. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4\,(2x^2+2ax+4b$

$4x^4$

$4x^2+2ax$ $\quad 8ax^3+4a^2x^2$

$\quad 8ax^3+4a^2x^2$

$4x^2+4ax+4b^2$ $\quad 16b^2x^2+16ab^2x+16b^4$

$\quad 16b^2x^2+16ab^2x+16b^4$

17.

$$\begin{array}{l|l}
 & 9-24x+58x^2-116x^3+129x^4-140x^5+100x^6\,(3-4x+7x^2-10x^3 \\
 & 9 \\ \hline
6-4x & -24x+58x^2 \\
 & -24x+16x^2 \\ \hline
6-8x+7x^2 & 42x^2-116x^3+129x^4 \\
 & 42x^2-56x^3+49x^4 \\ \hline
6-8x+14x^2-10x^3 & -60x^3+80x^4-140x^5+100x^6 \\
 & -60x^3+80x^4-140x^5+100x^6 \\ \hline
\end{array}$$

18.

$$\begin{array}{l|l}
 & 16a^4-40a^3b+25a^2b^2-80ab^2x+64b^2x^2+64a^2bx\,(4a^2-5ab+8bx \\
 & 16a^4 \\ \hline
8a^2-5ab & -40a^3b+25a^2b^2 \\
 & -40a^3b+25a^2b^2 \\ \hline
8a^2-10ab+8bx & 64a^2bx-80ab^2x+64b^2x^2 \\
 & 64a^2bx-80ab^2x+64b^2x^2 \\ \hline
\end{array}$$

19.

$$\begin{array}{l|l}
 & 9a^4-24a^3p^3-30a^2t+16a^2p^6+40ap^3t+25t^2\,(3a^2-4ap^3-5t \\
 & 9a^4 \\ \hline
6a^2-4ap^3 & -24a^3p^3+16a^2p^6 \\
 & -24a^3p^3+16a^2p^6 \\ \hline
6a^2-8ap^3-5t & -30a^2t+40ap^3t+25t^2 \\
 & -30a^2t+40ap^3t+25t^2 \\ \hline
\end{array}$$

20.

$$\begin{array}{l|l}
 & 4y^4x^2-12y^3x^3+17y^2x^4-12yx^5+4x^6\,(2y^2x-3yx^2+2x^3 \\
 & 4y^4x^2 \\ \hline
4y^2x-3yx^2 & -12y^3x^3+17y^2x^4 \\
 & -12y^3x^3+9y^2x^4 \\ \hline
4y^2x-6yx^2+2x^3 & 8y^2x^4-12yx^5+4x^6 \\
 & 8y^2x^4-12yx^5+4x^6 \\ \hline
\end{array}$$

21.

$$\begin{array}{l|l}
 & 25x^4y^2 - 30x^3y^3 + 29x^2y^4 - 12xy^5 + 4y^6\,(5x^2y - 3xy^2 + 2y^3 \\
 & 25x^4y^2 \\ \cline{2-2}
10x^2y - 3xy^2 & -30x^3y^3 + 29x^2y^4 \\
 & -30x^3y^3 + 9x^2y^2 \\ \cline{2-2}
10x^2y - 6xy^2 + 2y^3 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\
 & \qquad\qquad 20x^2y^4 - 12xy^5 + 4y^6 \\ \cline{2-2}
\end{array}$$

22.

$$\begin{array}{l|l}
 & 16x^4 - 24x^3y + 25x^2y^2 - 12xy^3 + 4y^4\,(4x^2 - 3xy + 2y^2 \\
 & 16x^4 \\ \cline{2-2}
8x^2 - 3xy & -24x^3y + 25x^2y^2 \\
 & -24x^3y + 9x^2y^2 \\ \cline{2-2}
8x^2 - 6xy + 2y^2 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\
 & \qquad\qquad 16x^2y^2 - 12xy^3 + 4y^4 \\ \cline{2-2}
\end{array}$$

23.

$$\begin{array}{l|l}
 & 9a^2 - 12ab + 24ac - 16bc + 4b^2 + 16c^2\,(3a - 2b + 4c \\
 & 9a^2 \\ \cline{2-2}
6a - 2b & -12ab + 4b^2 \\
 & -12ab + 4b^2 \\ \cline{2-2}
6a - 4b + 4c & \qquad\qquad 24ac - 16bc + 16c^2 \\
 & \qquad\qquad 24ac - 16bc + 16c^2 \\ \cline{2-2}
\end{array}$$

24.

$$\begin{array}{l|l}
 & x^4 + 19x^2 + 25 - 6x^3 - 30x\,(x^2 - 3x + 5 \\
 & x^4 \\ \cline{2-2}
2x^2 - 3x & -6x^3 + 19x^2 \\
 & -6x^3 + 9x^2 \\ \cline{2-2}
2x^2 - 6x + 5 & \qquad\qquad 10x^2 - 30x + 25 \\
 & \qquad\qquad 10x^2 - 30x + 25 \\ \cline{2-2}
\end{array}$$

25.

$$\begin{array}{l|l}
 & 25x^2 - 20xy + 4y^2 + 9z^2 - 12yz + 30xz \,(5x - 2y + 3z \\
 & 25x^2 \\ \hline
10x - 2y & -20xy + 4y^2 \\
 & -20xy + 4y^2 \\ \hline
10x - 4y + 3z & 30xz - 12yz + 9z^2 \\
 & 30xz - 12yz + 9z^2 \\ \hline
\end{array}$$

26.

$$\begin{array}{l|l}
 & 4x^4 - 4x^2y + 4x^2y^2 + y^4 - 2y^3 + y^2 \,(2x^2 - y + y^2 \\
 & 4x^4 \\ \hline
4x^2 - y & -4x^2y + y^2 \\
 & -4x^2y + y^2 \\ \hline
4x^2 - 2y + y^2 & 4x^2y^2 - 2y^3 + y^4 \\
 & 4x^2y^2 - 2y^3 + y^4 \\ \hline
\end{array}$$

LXXXI.

1.

$$\begin{array}{l|l}
 & 4a^6 + \frac{a^2b^4}{16} - a^4b^2 \,(2a^3 - \frac{ab^2}{4} \\
 & 4a^6 \\ \hline
4a^3 - \frac{ab^2}{4} & -a^4b^2 + \frac{a^2b^4}{16} \\
 & -a^4b^2 + \frac{a^2b^4}{16} \\ \hline
\end{array}$$

2.

$$\begin{array}{l|l}
 & \frac{9}{a^2} - 2 + \frac{a^2}{9} \left(\frac{3}{a} - \frac{a}{3}\right. \\
 & \frac{9}{a^2} \\ \hline
\frac{6}{a} - \frac{a}{3} & -2 + \frac{a^2}{9} \\
 & -2 + \frac{a^2}{9} \\ \hline
\end{array}$$

3.
$$a^4-2+\frac{1}{a^4}\left(a^2-\frac{1}{a^2}\right.$$
$$\begin{array}{r|l} & a^4 \\ \cline{2-2} 2a^2-\frac{1}{a^2} & -2+\frac{1}{a^4} \\ & -2+\frac{1}{a^4} \\ \cline{2-2} \end{array}$$

4.
$$\frac{a^2}{b^2}+2+\frac{b^2}{a^2}\left(\frac{a}{b}+\frac{b}{a}\right.$$
$$\begin{array}{r|l} & \frac{a^2}{b^2} \\ \cline{2-2} \frac{2a}{b}+\frac{b}{a} & 2+\frac{b^2}{a^2} \\ & 2+\frac{b^2}{a^2} \\ \cline{2-2} \end{array}$$

5.
$$x^4-2x^3+2x^2-x+\frac{1}{4}\left(x^2-x+\frac{1}{2}\right.$$
$$\begin{array}{r|l} & x^4 \\ \cline{2-2} 2x^2-x & -2x^3+2x^2 \\ & -2x^3+x^2 \\ \cline{2-2} 2x^2-2x+\frac{1}{2} & x^2-x+\frac{1}{4} \\ & x^2-x+\frac{1}{4} \\ \cline{2-2} \end{array}$$

6.
$$x^4+2x^3-x+\frac{1}{4}\left(x^2+x-\frac{1}{2}\right.$$
$$\begin{array}{r|l} & x^4 \\ \cline{2-2} 2x^2+x & 2x^3-x \\ & 2x^3+x^2 \\ \cline{2-2} 2x^2+2x-\frac{1}{2} & -x^2-x+\frac{1}{4} \\ & -x^2-x+\frac{1}{4} \\ \cline{2-2} \end{array}$$

7.

$$4a^2 - 12ab + ab^2 + 9b^2 - \frac{3b^3}{2} + \frac{b^4}{16}\Big(2a - 3b + \frac{b^2}{4}$$

$$\begin{array}{l|l}
 & 4a^2 \\ \hline
4a - 3b & -12ab + ab^2 + 9b^2 \\
 & -12ab + 9b^2 \\ \hline
4a - 6b + \frac{b^2}{4} & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\
 & ab^2 - \frac{3b^3}{2} + \frac{b^4}{16} \\ \hline
\end{array}$$

8.

$$x^4 + 8x^2 + 24 + \frac{16}{x^4} + \frac{32}{x^2}\Big(x^2 + 4 + \frac{4}{x^2}$$

$$\begin{array}{l|l}
 & x^4 \\ \hline
2x^2 + 4 & 8x^2 + 24 \\
 & 8x^2 + 16 \\ \hline
2x^2 + 8 + \frac{4}{x^2} & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\
 & 8 + \frac{32}{x^2} + \frac{16}{x^4} \\ \hline
\end{array}$$

9.

$$\frac{16}{9}a^6x^2 + \frac{16}{3}a^5x + 4a^4 - 2a^3x - 3a^2 + \frac{9}{16}\Big(\frac{4a^3x}{3} + 2a^2 - \frac{3}{4}$$

$$\begin{array}{l|l}
 & \frac{16}{9}a^6x^2 \\ \hline
\frac{8a^3x}{3} + 2a^2 & \frac{16}{3}a^5x + 4a^4 \\
 & \frac{16}{3}a^5x + 4a^4 \\ \hline
\frac{8a^3x}{3} + 4a^2 - \frac{3}{4} & -2a^3x - 3a^2 + \frac{9}{16} \\
 & -2a^3x - 3a^2 + \frac{9}{16} \\ \hline
\end{array}$$

10.

$$
\begin{array}{l|l}
 & \frac{1}{x^2}-\frac{4}{xy}+\frac{4}{y^2}+\frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2}\Big(\frac{1}{x}-\frac{2}{y}+\frac{3}{z} \\
 & \frac{1}{x^2} \\
\hline
\frac{2}{x}-\frac{2}{y} & -\frac{4}{xy}+\frac{4}{y^2} \\
 & -\frac{4}{xy}+\frac{4}{y^2} \\
\hline
\frac{2}{x}-\frac{4}{y}+\frac{3}{z} & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\
 & \frac{6}{xz}-\frac{12}{yz}+\frac{9}{z^2} \\
\hline
\end{array}
$$

11.

$$
\begin{array}{l|l}
 & 86m^2-\frac{48m}{n}+\frac{16}{n^2}+\frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25}\Big(6m-\frac{4}{n}+\frac{p}{5} \\
 & 36m^2 \\
\hline
12m-\frac{4}{n} & -\frac{48m}{n}+\frac{16}{n^2} \\
 & -\frac{48m}{n}+\frac{16}{n^2} \\
\hline
12m-\frac{8}{n}+\frac{p}{5} & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\
 & \frac{12mp}{5}-\frac{8p}{5n}+\frac{p^2}{25} \\
\hline
\end{array}
$$

12.

$$
\begin{array}{l|l}
 & a^2b^2-6abcd+9c^2d^2+\frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49}\Big(ab-3cd+\frac{ef}{7} \\
 & a^2b^2 \\
\hline
2ab-3cd & -6abcd+9c^2d^2 \\
 & -6abcd+9c^2d^2 \\
\hline
2ab-6cd+\frac{ef}{7} & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\
 & \frac{2abef}{7}-\frac{6cdef}{7}+\frac{e^2f^2}{49} \\
\hline
\end{array}
$$

13.

$$\begin{array}{ll}
 & \frac{4x^2}{z^2} - \frac{12xy}{z^2} + \frac{9y^2}{z^2} + 4 - \frac{6y}{x} + \frac{z^2}{x^2}\Big(\frac{2x}{z} - \frac{3y}{z} + \frac{z}{x} \\
 & \frac{4x^2}{z^2} \\ \hline
\frac{4x}{z} - \frac{3y}{z} & - \frac{12xy}{z^2} + \frac{9y^2}{z^2} \\
 & - \frac{12xy}{z^2} + \frac{9y^2}{z^2} \\ \hline
\frac{4x}{z} - \frac{3y}{z} + \frac{z}{x} & 4 - \frac{6y}{x} + \frac{z^2}{x^2} \\
 & 4 - \frac{6y}{x} + \frac{z^2}{x^2} \\ \hline
\end{array}$$

14.

$$\begin{array}{ll}
 & \frac{4m^2}{n^2} - \frac{16m}{n} + 4 + \frac{24n}{m} + \frac{9n^2}{m^2}\Big(\frac{2m}{n} - 4 - \frac{3n}{m} \\
 & \frac{4m^2}{n^2} \\ \hline
\frac{4m}{n} - 4 & - \frac{16m}{n} + 4 \\
 & - \frac{16m}{n} + 16 \\ \hline
\frac{4m}{n} - 8 - \frac{3n}{m} & - 12 + \frac{24n}{m} + \frac{9n^2}{m^2} \\
 & - 12 + \frac{24n}{m} + \frac{9n^2}{m^2} \\ \hline
\end{array}$$

15.

$$\begin{array}{l|l}
 & \frac{a^2}{9}-\frac{ab}{6}+\frac{b^2}{16}+\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}\Big(\frac{a}{3}-\frac{b}{4}+\frac{c}{5}-\frac{d}{2} \\
 & \frac{a^2}{9} \\
\hline
\frac{2a}{3}-\frac{b}{4} & -\frac{ab}{6}+\frac{b^2}{16} \\
 & -\frac{ab}{6}+\frac{b^2}{16} \\
\hline
\frac{2a}{3}-\frac{b}{2}+\frac{c}{5} & \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25} \\
 & \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25} \\
\hline
\frac{2a}{3}-\frac{b}{2}+\frac{2c}{5}-\frac{d}{2} & -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4} \\
 & -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4} \\
\hline
\end{array}$$

16.

$$\begin{array}{l|l}
 & 49x^4-28x^3-17x^2+6x+\frac{9}{4}\Big(7x^2-2x-\frac{3}{2} \\
 & 49x^4 \\
\hline
14x^2-2x & -28x^3-17x^2 \\
 & -28x^3+4x^2 \\
\hline
14x^2-4x-\frac{3}{2} & -21x^2+6x+\frac{9}{4} \\
 & -21x^2+6x+\frac{9}{4} \\
\hline
\end{array}$$

17.

$$\begin{array}{l|l}
 & 9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2\Big(3x^2-\frac{ax}{2}+bx \\
 & 9x^4 \\
\hline
6x^2-\frac{ax}{2} & -3ax^3+\frac{a^2x^2}{4} \\
 & -3ax^3+\frac{a^2x^2}{4} \\
\hline
6x^2-ax+bx & 6bx^3-abx^2+b^2x^2 \\
 & 6bx^3-abx^2+b^2x^2 \\
\hline
\end{array}$$

13.

$$
\begin{array}{ll}
 & \frac{4x^2}{z^2} - \frac{12xy}{z^2} + \frac{9y^2}{z^2} + 4 - \frac{6y}{x} + \frac{z^2}{x^2}\left(\frac{2x}{z} - \frac{3y}{z} + \frac{z}{x}\right. \\
 & \frac{4x^2}{z^2} \\
\hline
\frac{4x}{z} - \frac{3y}{z} & -\frac{12xy}{z^2} + \frac{9y^2}{z^2} \\
 & -\frac{12xy}{z^2} + \frac{9y^2}{z^2} \\
\hline
\frac{4x}{z} - \frac{3y}{z} + \frac{z}{x} & 4 - \frac{6y}{x} + \frac{z^2}{x^2} \\
 & 4 - \frac{6y}{x} + \frac{z^2}{x^2} \\
\hline
\end{array}
$$

14.

$$
\begin{array}{ll}
 & \frac{4m^2}{n^2} - \frac{16m}{n} + 4 + \frac{24n}{m} + \frac{9n^2}{m^2}\left(\frac{2m}{n} - 4 - \frac{3n}{m}\right. \\
 & \frac{4m^2}{n^2} \\
\hline
\frac{4m}{n} - 4 & -\frac{16m}{n} + 4 \\
 & -\frac{16m}{n} + 16 \\
\hline
\frac{4m}{n} - 8 - \frac{3n}{m} & -12 + \frac{24n}{m} + \frac{9n^2}{m^2} \\
 & -12 + \frac{24n}{m} + \frac{9n^2}{m^2} \\
\hline
\end{array}
$$

15.
$$\frac{a^2}{9}-\frac{ab}{6}+\frac{b^2}{16}+\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

$$\frac{a^2}{9} \qquad\qquad \left(\frac{a}{3}-\frac{b}{4}+\frac{c}{5}-\frac{d}{2}\right.$$

$$\frac{2a}{3}-\frac{b}{4} \quad \Big\backslash \quad -\frac{ab}{6}+\frac{b^2}{16}$$

$$-\frac{ab}{6}+\frac{b^2}{16}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{c}{5} \quad \Big\backslash \quad \frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}$$

$$\frac{2ac}{15}-\frac{bc}{10}+\frac{c^2}{25}$$

$$\frac{2a}{3}-\frac{b}{2}+\frac{2c}{5}-\frac{d}{2} \quad \Big\backslash \quad -\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

$$-\frac{ad}{3}+\frac{bd}{4}-\frac{cd}{5}+\frac{d^2}{4}$$

16.
$$49x^4-28x^3-17x^2+6x+\frac{9}{4}\left(7x^2-2x-\frac{3}{2}\right.$$

$$49x^4$$

$$14x^2-2x \quad \Big\backslash \quad -28x^3-17x^2$$

$$-28x^3+4x^2$$

$$14x^2-4x-\frac{3}{2} \quad \Big\backslash \quad -21x^2+6x+\frac{9}{4}$$

$$-21x^2+6x+\frac{9}{4}$$

17.
$$9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2\left(3x^2-\frac{ax}{2}+bx\right.$$

$$9x^4$$

$$6x^2-\frac{ax}{2} \quad \Big\backslash \quad -3ax^3+\frac{a^2x^2}{4}$$

$$-3ax^3+\frac{a^2x^2}{4}$$

$$6x^2-ax+bx \quad \Big\backslash \quad 6bx^3-abx^2+b^2x^2$$

$$6bx^3-abx^2+b^2x^2$$

18.

$$\begin{array}{r|l}
 & 9x^4 - 2x^3 - \frac{161}{9}x^2 + 2x + 9\ \left(3x^2 - \frac{x}{3} - 3\right. \\
 & 9x^4 \\ \hline
6x^2 - \frac{x}{3} & -2x^3 - \frac{161}{9}x^2 \\
 & -2x^3 + \frac{x^2}{9} \\ \hline
6x^2 - \frac{2x}{3} - 3 & -18x^2 + 2x + 9 \\
 & -18x^2 + 2x + 9 \\ \hline
\end{array}$$

LXXXIII.

1.

$$\begin{array}{r|l}
 & a^3 - 3a^2b + 3ab^2 - b^3\ (a - b \\
 & a^3 \\ \hline
3a^2 - 3ab + b^2 & -3a^2b + 3ab^2 - b^3 \\
 & -3a^2b + 3ab^2 - b^3 \\ \hline
\end{array}$$

2.

$$\begin{array}{r|l}
 & 8a^3 + 12a^2 + 6a + 1\ (2a + 1 \\
 & 8a^3 \\ \hline
12a^2 + 6a + 1 & 12a^2 + 6a + 1 \\
 & 12a^2 + 6a + 1 \\ \hline
\end{array}$$

3.

$$\begin{array}{r|l}
 & a^3 + 24a^2b + 192ab^2 + 512b^3\ (a + 8b \\
 & a^3 \\ \hline
3a^2 + 24ab + 64b^2 & 24a^2b + 192ab^2 + 512b^3 \\
 & 24a^2b + 192ab^2 + 512b^3 \\ \hline
\end{array}$$

4.

$$\begin{array}{r|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3\ (a + b + c \\
 & a^3 \\ \hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\
 & 3a^2b + 3ab^2 + b^3 \\ \hline
3a^2 + 6ab + 3b^2 & 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \\
+ 3ac + 3bc + c^2 & 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3 \\ \hline
\end{array}$$

5.

$$\begin{array}{r|l}
 & x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3\ (x - y + z \\
 & x^3 \\ \hline
3x^2 - 3xy + y^2 & -3x^2y + 3xy^2 - y^3 \\
 & -3x^2y + 3xy^2 - y^3 \\ \hline
3x^2 - 6xy + 3y^2 & 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 \\
+ 3xz - 3yz + z^2 & 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3 \\ \hline
\end{array}$$

6.

$$\begin{array}{l|l}
 & 27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1\,(3x^2 - 2x + 1 \\
 & 27x^6 \\ \hline
27x^4 - 18x^3 + 4x^2 & -54x^5 + 63x^4 - 44x^3 \\
 & -54x^5 + 36x^4 - 8x^3 \\ \hline
27x^4 - 36x^3 + 21x^2 - 6x + 1 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\
 & 27x^4 - 36x^3 + 21x^2 - 6x + 1 \\ \hline
\end{array}$$

7.

$$\begin{array}{l|l}
 & 1 - 3a + 6a^2 - 7a^3 + 6a^4 - 3a^5 + a^6\,(1 - a + a^2 \\
 & 1 \\ \hline
3 - 3a + a^2 & -3a + 6a^2 - 7a^3 \\
 & -3a + 3a^2 - a^3 \\ \hline
3 - 6a + 6a^2 - 3a^3 + a^4 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\
 & 3a^2 - 6a^3 + 6a^4 - 3a^5 + a^6 \\ \hline
\end{array}$$

8.

$$\begin{array}{l|l}
 & x^3 - 3x^2y + 3xy^2 - y^3 + 8z^3 + 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 \\
 & x^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x - y + 2z \\ \hline
3x^2 - 3xy + y^2 & -3x^2y + 3xy^2 - y^3 \\
 & -3x^2y + 3xy^2 - y^3 \\ \hline
3x^2 - 6xy + 3y^2 & 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 + 8z^3 \\
\quad + 6xz - 6yz + 4z^2 & 6x^2z - 12xyz + 6y^2z + 12xz^2 - 12yz^2 + 8z^3 \\ \hline
\end{array}$$

9.

$$\begin{array}{l|l}
 & a^6 - 12a^5 + 54a^4 - 112a^3 + 108a^2 - 48a + 8\,(a^2 - 4a + 2 \\
 & a^6 \\ \hline
3a^4 - 12a^3 + 16a^2 & -12a^5 + 54a^4 - 112a^3 \\
 & -12a^5 + 48a^4 - 64a^3 \\ \hline
3a^4 - 24a^3 + 54a^2 - 24a + 8 & 6a^4 - 48a^3 + 108a^2 - 48a + 8 \\
 & 6a^4 - 48a^3 + 108a^2 - 48a + 8 \\ \hline
\end{array}$$

10.

$$\begin{array}{l|l}
 & 8m^6-36m^5+66m^4-63m^3+33m^2-9m+1\,(2m^2-3m+1 \\
 & 8m^6 \\ \hline
12m^4-18m^3+9m^2 & -36m^5+66m^4-63m^3 \\
 & -36m^5+54m^4-27m^3 \\ \hline
12m^4-36m^3+33m^2 & 12m^4-36m^3+33m^2-9m+1 \\
\quad -9m+1 & 12m^4-36m^3+33m^2-9m+1 \\ \hline
\end{array}$$

11.

$$\begin{array}{l|l}
 & x^3+6x^2y+12xy^2+8y^3-3x^2z-12xyz-12y^2z+3xz^2+6yz^2-z^3 \\
 & x^3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x+2y-z \\ \hline
3x^2+6xy+4y^2 & 6x^2y+12xy^2+8y^3 \\
 & 6x^2y+12xy^2+8y^3 \\ \hline
3x^2+12xy+12y^2 & -3x^2z-12xyz-12y^2z+3xz^2+6yz^2-z^3 \\
\quad -3xz-6yz+z^2 & -3x^2z-12xyz-12y^2z+3xz^2+6yz^2-z^3 \\ \hline
\end{array}$$

12.

$$\begin{array}{l|l}
 & 8m^3-36m^2n+54mn^2-27n^3-12m^2r+36mnr-27n^2r+ \\
 & 8m^3 \qquad\qquad\qquad\qquad\qquad\qquad 6mr^2-9nr^2-r^3\,(2m-3n-r \\ \hline
12m^2-18mn+9n^2 & -36m^2n+54mn^2-27n^3 \\
 & -36m^2n+54mn^2-27n^3 \\ \hline
12m^2-36mn+27n^2- & -12m^2r+36mnr-27n^2r+6mr^2-9nr^2-r^3 \\
\quad 6mr-9nr+r^2 & -12m^2r+36mnr-27n^2r+6mr^2-9nr^2-r^3 \\ \hline
\end{array}$$

13.

$$\begin{array}{l|l}
 & m^3+3m^2-5+\frac{3}{m^2}-\frac{1}{m^3}\,(m+1-\frac{1}{m} \\
 & m^3 \\ \hline
3m^2+3m+1 & 3m^2-5+\frac{3}{m^2} \\
 & 3m^2+3m+1 \\ \hline
3m^2+6m-\frac{3}{m}+\frac{1}{m^2} & -3m-6+\frac{3}{m^2}-\frac{1}{m^3} \\
 & -3m-6+\frac{3}{m^2}-\frac{1}{m^3} \\ \hline
\end{array}$$

LXXXIV.

1

$$\begin{array}{ll}
 & 16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4\,(4a^2 - 12ax + 9x^2 \\
 & 16a^4 \\ \cline{2-2}
8a^2 - 12ax & -96a^3x + 216a^2x^2 \\
 & -96a^3x + 144a^2x^2 \\ \cline{2-2}
8a^2 - 24ax + 9x^2 & \qquad 72a^2x^2 - 216ax^3 + 81x^4 \\
 & \qquad 72a^2x^2 - 216ax^3 + 81x^4 \\ \cline{2-2}
\end{array}$$

and the square root of $4a^2 - 12ax + 9x^2$ is $2a - 3x$.

2.

$$\begin{array}{ll}
 & 1 - 8a + 24a^2 - 32a^3 + 16a^4\,(1 - 4a + 4a^2 \\
 & 1 \\ \cline{2-2}
2 - 4a & -8a + 24a^2 \\
 & -8a + 16a^2 \\ \cline{2-2}
2 - 8a + 4a^2 & \qquad 8a^2 - 32a^3 + 16a^4 \\
 & \qquad 8a^2 - 32a^3 + 16a^4 \\ \cline{2-2}
\end{array}$$

and the square root of $1 - 4a + 4a^2$ is $1 - 2a$.

3.

$$\begin{array}{ll}
 & 625 + 2000x + 2400x^2 + 1280x^3 + 256x^4\,(25 + 40x + 16x^2 \\
 & 625 \\ \cline{2-2}
50 + 40x & 2000x + 2400x^2 \\
 & 2000x + 1600x^2 \\ \cline{2-2}
50 + 80x + 16x^2 & \qquad 800x^2 + 1280x^3 + 256x^4 \\
 & \qquad 800x^2 + 1280x^3 + 256x^4 \\ \cline{2-2}
\end{array}$$

and the square root of $25 + 40x + 16x^2$ is $5 + 4x$.

4.

$$\begin{array}{l|l}
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6(a^3-3a^2b+3ab^2-b^3 \\
 & a^6 \\ \hline
2a^3-3a^2b & -6a^5b+15a^4b^2 \\
 & -6a^5b+9a^4b^2 \\ \hline
2a^3-6a^2b+3ab^2 & 6a^4b^2-20a^3b^3+15a^2b^4 \\
 & 6a^4b^2-18a^3b^3+9a^2b^4 \\ \hline
2a^3-6a^2b+6ab^2-b^3 & -2a^3b^3+6a^2b^4-6ab^5+b^6 \\
 & -2a^3b^3+6a^2b^4-6ab^5+b^6 \\ \hline
\end{array}$$

and the cube root of $a^3-3a^2b+3ab^2-b^3$ is $a-b$.

5.

$$\begin{array}{l|l}
 & x^6+6x^5+15x^4+20x^3+15x^2+6x+1\,(x^3+3x^2+3x+1 \\
 & x^6 \\ \hline
2x^3+3x^2 & 6x^5+15x^4 \\
 & 6x^5+9x^4 \\ \hline
2x^3+6x^2+3x & 6x^4+20x^3+15x^2 \\
 & 6x^4+18x^3+9x^2 \\ \hline
2x^3+6x^2+6x+1 & 2x^3+6x^2+6x+1 \\
 & 2x^3+6x^2+6x+1 \\ \hline
\end{array}$$

and the cube root of x^3+3x^2+3x+1 is $x+1$.

6.

$$\begin{array}{l|l}
 & m^6-12m^5+60m^4-160m^3+240m^2-192m+64\,(m^3-6m^2+12m-8 \\
 & m^6 \\ \hline
2m^3-6m^2 & -12m^5+60m^4 \\
 & -12m^5+36m^4 \\ \hline
2m^3-12m^2+12m & 24m^4-160m^3+240m^2 \\
 & 24m^4-144m^3+144m^2 \\ \hline
2m^3-12m^2+24m-8 & -16m^3+96m^2-192m+64 \\
 & -16m^3+96m^2-192m+64 \\ \hline
\end{array}$$

and the cube root of $m^3-6m^2+12m-8$ is $m-2$.

LXXXV.

1. $x=\pm 8$. 2. $x=\pm ab$. 3. $x^2=10000$; $x=\pm 100$.

4. $x^2=49$; $x=\pm 7$. 5. $3x^2=33$; $x^2=11$; $x=\pm\sqrt{11}$.

6. $x^2=64a^4c^6$, etc. 7. $4x^2-48=3x^2-12$; $x^2=36$, etc.

8. $250000-x^2=233359$; $x^2=16641$; $x=\pm 129$.

9. $8112=3x^2$; $x^2=2704$, etc.

10. $\frac{11x^2}{2}-18x+65=9x^2-18x+9$; $11x^2-18x^2=-112$; $7x^2=112$, etc.

11. $mx^2=q-n$; $x^2=\frac{q-n}{m}$, etc.

12. $x^2-ax+b=ax^2-ax$; $ax^2-x^2=b$; $x^2=\frac{b}{a-1}$, etc.

13. $180x^2-225=114x^2+171$; $66x^2=396$; $x^2=6$, etc.

14. $42x^2-126=35x^2-70$; $7x^2=56$; $x^2=8$; $x^2=4\times 2$; $x=\pm 2\sqrt{2}$.

LXXXVI.

1. $x^2+6x+9=81$; $x+3=\pm 9$, etc.
2. $x^2+12x+36=100$; $x+6=\pm 10$, etc.
3. $x^2+14x+49=64$; $x+7=\pm 8$, etc.
4. $x^2+46x+529=625$; $x+23=\pm 25$, etc.
5. $x^2+128x+4096=4489$; $x+64=\pm 67$, etc.
6. $x^2+8x+16=81$; $x+4=\pm 9$, etc.
7. $x^2+18x+81=324$; $x+9=\pm 18$, etc.
8. $x^2+16x+64=484$; $x+8=\pm 22$, etc.

LXXXVII.

1. $x^2-6x+9=16$; $x-3=\pm 4$, etc.

2. $x^2-4x+4=9$; $x-2=\pm 3$, etc.

3. $x^2-20x+100=121$; $x-10=\pm 11$, etc.

4. $x^2-2x+1=64$; $x-1=\pm 8$, etc.

5. $x^2-12x+36=4$; $x-6=\pm 2$, etc.

6. $x^2-14x+49=4$; $x-7=\pm 2$, etc.

7. $x^2-234x+13689=1$; $x-117=\pm 1$, etc.

8. $x^2-5x+6=15x+42$; $x^2-20x=36$; $x^2-20x+100=136$;
$x-10=\pm\sqrt{4\times 34}$; $x=10\pm 2\sqrt{34}$.

9. $3x^2-17x-2x^2-5x+120=0$; $x^2-22x+121=1$, etc.

10. $x^2-10x+25+x^2-14x+49=x^2-8x+46$; $x^2-16x=-28$, etc.

LXXXVIII.

1. $x^2+7x+\frac{49}{4}=\frac{169}{4}$; $x+\frac{7}{2}=\pm\frac{13}{2}$, etc.

2. $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

3. $x^2+9x+\frac{81}{4}=\frac{256}{4}$; $x+\frac{9}{2}=\pm\frac{16}{2}$, etc.

4. $x^2-13x+\frac{169}{4}=\frac{729}{4}$; $x-\frac{13}{2}=\pm\frac{27}{2}$, etc.

5. $x^2+x+\frac{1}{4}=\frac{9}{16}$; $x+\frac{1}{2}=\pm\frac{3}{4}$, etc.

6. $x^2-x+\frac{1}{4}=\frac{289}{4}$; $x-\frac{1}{2}=\pm\frac{17}{2}$, etc.

7. $x^2+37x+\frac{1369}{4}=\frac{16129}{4}$; $x+\frac{37}{2}=\pm\frac{127}{2}$, etc.

8. $x^2-x+\frac{1}{4}=\frac{225}{4}$; $x-\frac{1}{2}=\pm\frac{15}{2}$, etc.

9. $5x-x^2+2x^2-14x-10x+60=0$; $x^2-19x=-60$;

$x^2-19x+\frac{361}{4}=\frac{121}{4}$, etc.

10. $35x^2-312x+693-34x^2+21x+45=448$; $x^2-291x=-290$;

$x^2-291x+\frac{84681}{4}=\frac{83521}{4}$; $x-\frac{291}{2}=\pm\frac{289}{2}$, etc.

LXXXIX.

1. $x^2-\frac{2}{3}x+\frac{1}{9}=\frac{36}{9}$; $x-\frac{1}{3}=\pm\frac{6}{3}$, etc.

2. $x^2+\frac{4}{5}x+\frac{4}{25}=\frac{1}{25}$; $x+\frac{2}{5}=\pm\frac{1}{5}$, etc.

3. $x^2-\frac{28x}{9}+\frac{196}{81}=\frac{169}{81}$; $x-\frac{14}{9}=\pm\frac{13}{9}$, etc.

4. $x^2-\frac{8}{11}x+\frac{16}{121}=\frac{49}{121}$; $x-\frac{4}{11}=\pm\frac{7}{11}$, etc.

5. $x^2+\frac{4}{35}x+\frac{4}{1225}=\frac{529}{1225}$; $x+\frac{2}{35}=\pm\frac{23}{35}$, etc.

6. $x^2-\frac{16}{5}x+\frac{64}{25}=\frac{144}{25}$; $x-\frac{8}{5}=\pm\frac{12}{5}$, etc.

7. $x^2-\frac{26}{3}x+\frac{169}{9}=\frac{121}{9}$; $x-\frac{13}{3}=\pm\frac{11}{3}$, etc.

8. $x^2-\frac{4}{7}x+\frac{4}{49}=\frac{2209}{49}$; $x-\frac{2}{7}=\pm\frac{47}{7}$, etc.

XC.

1. $x^2-\frac{1}{3}x+\frac{1}{36}=\frac{289}{36}$; $x-\frac{1}{6}=\pm\frac{17}{6}$, etc.

2. $x^2-\frac{1}{5}x+\frac{1}{100}=\frac{9801}{100}$; $x-\frac{1}{10}=\pm\frac{99}{10}$, etc.

3. $x^2+\frac{1}{2x}+\frac{1}{16}=\frac{625}{16}$; $x+\frac{1}{4}=\pm\frac{25}{4}$, etc.

4. $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{1225}{16}$; $x+\frac{3}{4}=\pm\frac{35}{4}$, etc.

5. $x^2-\frac{9}{5}x+\frac{81}{100}=\frac{1681}{100}$; $x-\frac{9}{10}=\pm\frac{41}{10}$, etc.

6. $x^2-\frac{11}{2}x+\frac{121}{16}=\frac{25}{16}$; $x-\frac{11}{4}=\pm\frac{5}{4}$, etc.

7. $x^2-\frac{15}{4}x+\frac{225}{64}=\frac{2401}{64}$; $x-\frac{15}{8}=\pm\frac{49}{8}$, etc.

8. $x^2-\frac{23}{7}x+\frac{529}{196}=\frac{676}{196}$; $x-\frac{23}{14}=\pm\frac{26}{14}$, etc.

XCI.

1. $x^2+2ax+a^2=2a^2$; $x+a=\pm\sqrt{2}.a$, etc.

2. $x^2-4ax+4a^2=11a^2$; $x-2a=\pm\sqrt{11}.a$, etc.

3. $x^2+3mx+\frac{9m^2}{4}=4m^2$; $x+\frac{3m}{2}=\pm 2m$, etc.

4. $x^2-\frac{5nx}{2}+\frac{25n^2}{16}=\frac{49n^2}{16}$; $x-\frac{5n}{4}=\pm\frac{7n}{4}$, etc.

5. $x^2+(a-1)x+\frac{(a-1)^2}{4}=\frac{a^2+2a+1}{4}$; $x+\frac{a-1}{2}=\pm\frac{a+1}{2}$, etc.

6. $x^2+(a-b)x+\frac{(a-b)^2}{4}=\frac{(a+b)^2}{4}$; $x+\frac{a-b}{2}=\pm\frac{a+b}{2}$, etc.

7. $\frac{a^2}{(x+a)^2}=\frac{b^2}{(x-a)^2}$; and, taking the square root of each side,

$\frac{a}{x+a}=\pm\frac{b}{x-a}$, etc.

8. $acx^2-adx+bcx=bd$; $x^2+\frac{bc-ad}{ac}x=\frac{bd}{ac}$;

$x^2+\frac{bc-ad}{ac}x+\frac{(bc-ad)^2}{4a^2c^2}=\frac{(bc+ad)^2}{4a^2c^2}$;

$x+\frac{bc-ad}{2ac}=\pm\frac{bc+ad}{2ac}$; $x=\frac{2ad}{ac}$ or $-\frac{2bc}{2ac}$, etc.

9. $(a+b)x^2-cx=\frac{ac}{a+b}$; $x^2-\frac{cx}{a+b}=\frac{ac}{(a+b)^2}$;

$x^2-\frac{cx}{a+b}+\frac{c^2}{4(a+b)^2}=\frac{c^2+4ac}{4(a+b)^2}$; $x-\frac{c}{2(a+b)}=\pm\frac{\sqrt{c^2+4ac}}{2(a+b)}$, etc.

10. $x^2-\frac{2b^2x}{ac}=-\frac{b^4}{a^2c^2}$; $x^2-\frac{2b^2x}{ac}+\frac{b^4}{a^2c^2}=0$; $x-\frac{b^2}{ac}=0$; $x=\frac{b^2}{ac}$.

11. $x^2+\frac{3a^2+b^2}{abc}x=\frac{6a^2+ab-2b^2}{abc^2}$;

$x^2+\frac{3a^2+b^2}{abc}x+\frac{(3a^2+b^2)^2}{4a^2b^2c^2}=\frac{24a^3b+4a^2b^2-8ab^3+9a^4+6a^2b^2+b^4}{4a^2b^2c^2}$

$x+\frac{3a^2+b^2}{2abc}=\pm\frac{3a^2+4ab-b^2}{2abc}$, etc.

12. $x^2 + \frac{4a^2c^2 + 4abd^2}{4a^2 - 9cd^2}x = -\frac{a^2c^4 + 2abc^2d^2 + b^2d^4}{4a^2 - 9cd^2}$;

$$x^2 + \frac{4a^2c^2 + 4abd^2}{4a^2 - 9cd^2}x + \frac{(2a^2c^2 + 2abd^2)^2}{(4a^2 - 9cd^2)^2} = \frac{9cd^2(a^2c^4 + 2abc^2d^2 + b^2d^4)}{(4a^2 - 9cd^2)^2};$$

$$x + \frac{2a^2c^2 + 2abd^2}{4a^2 - 9cd^2} = \pm\frac{3d\sqrt{c}.(ac^2 + bd^2)}{4a^2 - 9cd^2};$$

$$x = \frac{-2a(ac^2 + bd^2) \pm 3d\sqrt{c}.(ac^2 + bd^2)}{4a^2 - 9cd^2};$$

$$x = \frac{(-2a \pm 3d\sqrt{c})(ac^2 + bd^2)}{(2a + 3d\sqrt{c})(2a - 3d\sqrt{c})}, \text{ etc.}$$

XCII.

1. $x^2 - 7x = 8$; $x^2 - 7x + \frac{49}{4} = \frac{81}{4}$; $x - \frac{7}{2} = \pm\frac{9}{2}$, etc.

2. $x^2 - 5x = 6$; $x^2 - 5x + \frac{25}{4} = \frac{49}{4}$; $x - \frac{5}{2} = \pm\frac{7}{2}$, etc.

3. $x^2 - 11x = 12$; $x^2 - 11x + \frac{121}{4} = \frac{169}{4}$; $x - \frac{11}{2} = \pm\frac{13}{2}$, etc.

4. $x^2 - 13x = 14$; $x^2 - 13x + \frac{169}{4} = \frac{225}{4}$; $x - \frac{13}{2} = \pm\frac{15}{2}$, etc.

5. $x^2 + 7x = 18$; $x^2 + 7x + \frac{49}{4} = \frac{121}{4}$; $x + \frac{7}{2} = \pm\frac{11}{2}$, etc.

6. $4x^2 - 12x - 12 + x = 22x - 66$; $4x^2 - 33x = -54$; $x^2 - \frac{33x}{4} = -\frac{54}{4}$;

$$x^2 - \frac{33x}{4} + \frac{1089}{64} = \frac{225}{64};\ x - \frac{33}{8} = \pm\frac{15}{8}, \text{ etc.}$$

7. $x^2 - 9x + \frac{81}{4} = \frac{1}{4}$; $x - \frac{9}{2} = \pm\frac{1}{2}$, etc.

8. $10x^2 - 30x - 6x + 6 = 7x^2 - 27x + 18$; $3x^2 - 9x = 12$; $x^2 - 3x = 4$, etc.

9. $x^2-6x=16$; $x^2-6x+9=25$; $x-3=\pm 5$, etc.

10. $8x^2+20x-x^2+9=4x^2+22x+30$; $3x^2-2x=21$;

$$x^2-\frac{2x}{3}=7;\ x^2-\frac{2x}{3}+\frac{1}{9}=\frac{64}{9};\ x-\frac{1}{3}=\pm\frac{8}{3},\text{ etc.}$$

11. $8x^2+12x-x^2+49=4x^2+34x+42$; $3x^2-22x=-7$;

$$x^2-\frac{22x}{3}=-\frac{7}{3};\ x^2-\frac{22x}{3}+\frac{121}{9}=\frac{100}{9};\ x-\frac{11}{3}=\pm\frac{10}{3},\text{ etc.}$$

12. $x^2-11x=12$; $x^2-11x+\frac{121}{4}=\frac{169}{4}$; $x-\frac{11}{2}=\pm\frac{13}{2}$, etc.

13. $x^2-13x=14$; $x^2-13x+\frac{169}{4}=\frac{225}{4}$; $x-\frac{13}{2}=\pm\frac{15}{2}$, etc.

14. $12x^2-8x+177=192$; $x^2-\frac{2x}{3}=\frac{5}{4}$; $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{49}{36}$, etc.

15. $3x^2-169=26x$; $x^2-\frac{26x}{3}=\frac{169}{3}$; $x^2-\frac{26x}{3}+\frac{169}{9}=\frac{676}{9}$, etc.

16. $x^2-9x=-20$; $x^2-9x+\frac{81}{4}=\frac{1}{4}$; $x-\frac{9}{2}=\pm\frac{1}{2}$, etc.

17. $6x^2-28x-48-300+20x=7x^2-56x+84$; $x^2-48x=-432$, etc.

18. $x^2-8x=-12$; $x^2-8x+16=4$; $x-4=\pm 2$, etc.

19. $9x^2-90x+125-54x^2=9x^2-75x$; $54x^2+15x=125$;

$$x^2+\frac{15x}{54}=\frac{125}{54};\ x^2+\frac{15x}{54}+\frac{225}{11664}=\frac{27225}{11664};\ x+\frac{15}{108}=\pm\frac{165}{108},\text{ etc.}$$

20. $7x^2+35x-8x^2+20x=6x^2+16x-70$; $7x^2-39x=70$;

$$x^2-\frac{39x}{7}=10;\ x^2-\frac{39x}{7}+\frac{1521}{196}=\frac{3481}{196};\ x-\frac{39}{14}=\pm\frac{59}{14},\text{ etc.}$$

21. $8x^2-20x+6x^2+16x-70=7x^2+35x$; $7x^2-39x=70$, etc.

22. $x^2-6x+9+4x=44$; $x^2-2x=35$, etc.

23. $x^2+11x=7x^2-9-4x$; $6x^2-15x=9$; $2x^2-5x=3$; $x^2-\frac{5x}{2}=\frac{3}{2}$;
$x^2-\frac{5x}{2}+\frac{25}{16}=\frac{49}{16}$; $x-\frac{5}{4}=\pm\frac{7}{4}$, etc.

24. $x^2+\frac{x}{6}=\frac{2}{6}$; $x^2+\frac{x}{6}+\frac{1}{144}=\frac{49}{144}$; $x+\frac{1}{12}=\pm\frac{7}{12}$, etc.

25. $x^2-\frac{x}{2}+\frac{1}{16}=\frac{25}{144}$; $x-\frac{1}{4}=\pm\frac{5}{12}$, etc.

26. $x^2-x+\frac{1}{4}=\frac{841}{4}$; $x-\frac{1}{2}=\pm\frac{29}{2}$, etc.

27. $6x+2x+2=3x^2+3x$; $3x^2-5x=2$; $x^2-\frac{5x}{3}=\frac{2}{3}$, etc.

28. $4x^2-33=x$; $x^2-\frac{x}{4}=\frac{33}{4}$; $x^2-\frac{x}{4}+\frac{1}{64}=\frac{529}{64}$, etc.

29. $2x^2=3x^2-3x+2x^2-4x+2$; $x^2-\frac{7x}{3}=-\frac{2}{3}$, etc.

30. $x^2-\frac{7x}{15}=\frac{46}{15}$; $x^2-\frac{7x}{15}+\frac{49}{900}=\frac{2809}{900}$; $x-\frac{7}{30}=\pm\frac{53}{30}$, etc.

31. $5x+10-10x+20=3x^2-12$; $x^2+\frac{5x}{3}=14$, etc.

32. $4x^2-100+40x-4x^2=75x-15x^2$; $15x^2-35x=100$;
$x^2-\frac{7x}{3}=\frac{20}{3}$; $x^2-\frac{7x}{3}+\frac{49}{36}=\frac{289}{36}$, etc.

33. $90x-126+18x=22x^2$; $x^2-\frac{54x}{11}=-\frac{63}{11}$; $x^2-\frac{54x}{11}+\frac{729}{121}=\frac{36}{121}$, etc.

34. $3x^2-5x=7x+420$; $x^2-4x=140$, etc.

35. $\frac{48-12x+40-8x}{20-9x+x^2}=\frac{32}{x+2}$; $\frac{22-5x}{20-9x+x^2}=\frac{8}{x+2}$;
$44+12x-5x^2=160-72x+8x^2$; $13x^2-84x=-116$;
$x^2-\frac{84x}{13}=-\frac{116}{13}$; $x^2-\frac{84x}{13}+\frac{1764}{169}=\frac{256}{169}$, etc.

36. $10x^2+490-140x+10x^2=203x-29x^2$; $49x^2-343x=-490$;
$x^2-7x=-10$, etc.

37. $x^2+(a+b)x+\frac{(a+b)^2}{4}=\frac{(a-b)^2}{4}$; $x+\frac{a+b}{2}=\pm\frac{a-b}{2}$, etc.

38. $x^2-(b-a)x+\frac{(b-a)^2}{4}=\frac{(b+a)^2}{4}$; $x-\frac{b-a}{2}=\pm\frac{b+a}{2}$, etc.

39. $x^2-2ax+a^2=b^2$; $x-a=\pm b$, etc.

40. $x^2-(a^2-a^3)x+\frac{(a^2-a^3)^2}{4}=\frac{(a^2+a^3)^2}{4}$; $x-\frac{a^2-a^3}{2}=\pm\frac{a^2+a^3}{2}$, etc.

41. $x^2+\frac{ax}{b}+\frac{a^2}{4b^2}=\frac{9a^2}{4b^2}$; $x+\frac{a}{2b}=\pm\frac{3a}{2b}$, etc.

42. $x^2-\frac{a^2+b^2}{ab}x+\frac{(a^2+b^2)^2}{4a^2b^2}=\frac{(a^2-b^2)^2}{4a^2b^2}$; $x-\frac{a^2+b^2}{2ab}=\pm\frac{a^2-b^2}{2ab}$.

XCIII.

1. $\left.\begin{array}{ll} x^2+2xy+y^2 & =1600 \\ 4xy & =1200 \end{array}\right\}$, $x^2-2xy+y^2=400$, $\left\{\begin{array}{l} x+y=40 \\ x-y=\pm 20 \end{array}\right.$, etc.

2. $\left.\begin{array}{ll} x^2+2xy+y^2 & =169 \\ 4xy & =144 \end{array}\right\}$, $x^2-2xy+y^2=25$, $\left\{\begin{array}{l} x+y=13 \\ x-y=\pm\ 5 \end{array}\right.$, etc.

3. $\left.\begin{array}{ll} x^2+2xy+y^2 & =841 \\ 4xy & =400 \end{array}\right\}$, $x^2-2xy+y^2=441$, $\left\{\begin{array}{l} x+y=29 \\ x-y=\pm 21 \end{array}\right.$, etc.

4. $\left.\begin{array}{ll} x^2-2xy+y^2 & =361 \\ 4xy & =264 \end{array}\right\}$, $x^2+2xy+y^2=625$, $\left\{\begin{array}{l} x-y=19 \\ x+y=\pm 25 \end{array}\right.$, etc.

5. $\left.\begin{array}{ll} x^2-2xy+y^2 & =2025 \\ 4xy & =1000 \end{array}\right\}$, $x^2+2xy+y^2=3025$, $\left\{\begin{array}{l} x-y=45 \\ x+y=\pm 55 \end{array}\right.$, etc.

6. $\left.\begin{array}{ll} x^2-2xy+y^2 & =9801 \\ 4xy & =\ 400 \end{array}\right\}$, $x^2+2xy+y^2=10201$, $\left\{\begin{array}{l} x-y=99 \\ x+y=\pm 101 \end{array}\right.$, etc.

XCIV.

1. $\left.\begin{aligned} x^2-2xy+y^2&=16\\ x^2\qquad+y^2&=40\end{aligned}\right\}$; $2xy=24$; $x^2+2xy+y^2=64$, etc.

2. $\left.\begin{aligned} x^2-2xy+y^2&=100\\ x^2\qquad+y^2&=178\end{aligned}\right\}$; $2xy=78$; $x^2+2xy+y^2=256$, etc.

3. $\left.\begin{aligned} x^2-2xy+y^2&=196\\ x^2\qquad+y^2&=436\end{aligned}\right\}$; $2xy=240$; $x^2+2xy+y^2=676$, etc.

4. $\left.\begin{aligned} x^2+2xy+y^2&=64\\ x^2\qquad+y^2&=32\end{aligned}\right\}$; $2xy=32$; $x^2-2xy+y^2=0$, etc.

5. $\left.\begin{aligned} x^2+2xy+y^2&=144\\ x^2\qquad+y^2&=104\end{aligned}\right\}$; $2xy=40$; $x^2-2xy+y^2=64$, etc.

6. $\left.\begin{aligned} x^2+2xy+y^2&=2401\\ x^2\qquad+y^2&=1681\end{aligned}\right\}$; $2xy=720$; $x^2-2xy+y^2=961$, etc.

XCV.

1. $\left.\begin{aligned} x^2-xy+y^2&=13\\ x^2+2xy+y^2&=49\end{aligned}\right\}$; $3xy=36$; $x^2-2xy+y^2=1$, etc.

2. $\left.\begin{aligned} x^2-xy+y^2&=31\\ x^2+2xy+y^2&=121\end{aligned}\right\}$; $3xy=90$; $x^2-2xy+y^2=1$, etc.

3. $\left.\begin{aligned} x^2-xy+y^2&=84\\ x^2+2xy+y^2&=144\end{aligned}\right\}$; $3xy=60$; $x^2-2xy+y^2=64$, etc.

4. $\left.\begin{aligned} x^2+xy+y^2&=28\\ x^2-2xy+y^2&=4\end{aligned}\right\}$; $3xy=24$; $x^2+2xy+y^2=36$, etc.

5. $\left.\begin{aligned} x^2+xy+y^2&=49\\ x^2-2xy+y^2&=4\end{aligned}\right\}$; $3xy=45$; $x^2+2xy+y^2=64$, etc.

6. $\left.\begin{aligned} x^2+xy+y^2&=93\\ x^2-2xy+y^2&=9\end{aligned}\right\}$; $3xy=84$; $x^2+2xy+y^2=121$, etc.

XCVI.

1. $$\left.\begin{array}{l}\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{81}{400}\\[2ex] \dfrac{1}{x^2}\qquad\;\;+\dfrac{1}{y^2}=\dfrac{41}{400}\end{array}\right\};\ \frac{2}{xy}=\frac{40}{400};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{400},\text{ etc.}$$

2. $$\left.\begin{array}{l}\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=\dfrac{9}{16}\\[2ex] \dfrac{1}{x^2}\qquad\;\;+\dfrac{1}{y^2}=\dfrac{5}{16}\end{array}\right\};\ \frac{2}{xy}=\frac{4}{16};\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{16},\text{ etc.}$$

3. $$\left.\begin{array}{l}\dfrac{1}{x^2}+\dfrac{2}{xy}+\dfrac{1}{y^2}=25\\[2ex] \dfrac{1}{x^2}\qquad\;\;+\dfrac{1}{y^2}=13\end{array}\right\};\ \frac{2}{xy}=12;\ \frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1,\text{ etc.}$$

4. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{12}$, etc.

5. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=\frac{7}{2}$, etc.

6. Divide the second equation by the first; then $\frac{1}{x}-\frac{1}{y}=7$, etc.

XCVII.

1. $x-y=1$; $x^2-2xy+y^2=1$; $3xy=36$, etc.

2. $x^2+12xy+36y^2=576$; $x+6y=\pm 24$.

   Now, from the first equation, $x(x+6y)=144$, $\therefore\ \pm 24x=144$, etc.

3. $x^2+2xy+y^2=441$; $x+y=\pm 21$.

   Now, from the first equation, $x(x+y)=210$, $\therefore\ \pm 21x=210$, etc.

4. $x^2+2xy+y^2=100$; and $x^2-2xy+y^2=36$;

   hence $x+y=\pm 10$, and $x-y=\pm 6$, etc.

5. $x+y=8$; $x^2+2xy+y^2=64$; $3xy=45$, etc.

6. $4x=10+5y$; put $\frac{10+5y}{4}$ for x in the first equation ;

then $\frac{(10+5y)^2}{4}+\frac{90y+45y^2}{4}=100$; $70y^2+190y=660$;

$7y^2+19y=66$; whence $y=2$ or $-\frac{33}{7}$, etc.

7. Put $y=mx$. Then $\frac{x^2+mx^2+m^2x^2}{3m^2x^2-5mx^2}=\frac{39}{25}$; $\frac{1+m+m^2}{3m^2-5m}=\frac{39}{25}$;

$25+25m+25m^2=117m^2-195m$; $92m^2-220m=25$;

hence we get $m=\frac{5}{2}$ or $-\frac{5}{46}$. Taking the value $\frac{5}{2}$, we get

$x^2+\frac{5x^2}{2}+\frac{25x^2}{4}=39$; $39x^2=39\times 4$; $x=\pm 2$, etc.

8. Put $y=mx$. Then $\frac{x^2+mx^2}{mx^2-m^2x^2}=\frac{66}{5}$; $\frac{1+m}{m-m^2}=\frac{66}{5}$;

$5+5m=66m-66m^2$; $m^2-\frac{61m}{66}=-\frac{5}{66}$; hence we get

$m=\frac{1}{11}$, or $\frac{5}{6}$. Taking the value $\frac{5}{6}$, $x^2+\frac{5x^2}{6}=66$,

whence $x^2=36$, $x=\pm 6$, etc.

9. Put $y=mx$. Then $\frac{3+4m}{5m+2m^2}=\frac{20}{12}$; whence $m=\frac{1}{2}$ or $-\frac{3}{10}$.

Taking the value $\frac{1}{2}$, we get $3x^2+2x^2=20$; $x=\pm 2$, etc.

10. Put $y=mx$. Then $\frac{x^2-mx^2+m^2x^2}{3x^2+13mx^2+8m^2x^2}=\frac{7}{162}$; $\frac{1-m+m^2}{3+13m+8m^2}=\frac{7}{162}$

$162-162m+162m^2=21+91m+56m^2$; $106m^2-253m=-141$;

$m^2-\frac{253}{136}m=-\frac{141}{106}$; $m-\frac{253}{212}=\pm\frac{65}{212}$; $m=\frac{3}{2}$ or $-\frac{47}{53}$.

Taking the value $\frac{3}{2}$, we get $x^2-\frac{3x^2}{2}+\frac{9x^2}{4}=7$; $x=\pm 2$, etc.

11. Put $y=mx$. Then $\frac{1-m}{m+m^2}=\frac{35}{18}$; $35m^2+53m=18$;

$m+\frac{53}{70}=\pm\frac{73}{70}$; $m=\frac{2}{7}$ or $-\frac{9}{10}$.

Taking the value $\frac{2}{7}$ we get $x=\pm 7$, etc.

12. $x=\frac{29-7y}{5}$; putting this for x in the first equation, we get

$$\frac{2523-1218y+147y^2}{25}+\frac{116y-28y^2}{5}+5y^2=71;$$

$132y^2-638y=-748$; $y=2$ or $\frac{187}{66}$, etc.

13. $x+y=22$; $x^2+2xy+y^2=484$; $3xy=360$; $xy=120$;
$x^2-2xy+y^2=4$; $x-y=\pm 2$, etc.

14. Subtract the second equation from the first; $x^2+2xy+y^2=169$; $x+y=\pm 13$. Take the positive value, and for y put $13-x$ in the first equation; $x^2+117x-9x^2=340$; $x^2-\frac{117}{8}x=-\frac{340}{8}$;

$x^2-\frac{117}{8}x+\frac{13689}{256}=\frac{2809}{256}$; $x-\frac{117}{16}=\pm\frac{53}{16}$, etc.

15. $\left.\begin{array}{r}x^2+y^2=225\\ 2xy=216\end{array}\right\}$, $\left.\begin{array}{r}x^2+y^2+2xy=441\\ x^2+y^2-2xy=9\end{array}\right\}$, $\left.\begin{array}{r}x+y=\pm 21\\ x-y=\pm 3\end{array}\right\}$, etc.

XCVIII.

1. Let x be the number; then $\frac{x}{2}\times\frac{x}{3}=864$; $x^2=5184$, etc.

2. Let x be the number; then $\left(\frac{x}{7}\times\frac{x}{8}\right)\div 3=298\frac{2}{3}$; $\frac{x^2}{56}=896$, etc.

3. Let x be the number; then $(94-x)(94+x)=8512$, etc.

4. Let x be the greater ; then $\frac{750}{x}$ is the less ; and $x \times \frac{x}{750} = 3\frac{1}{3}$, etc.

5. Let x and y be the numbers ; then $x^2 + y^2 = 13001$ and $x^2 - y^2 = 1449$; adding, $2x^2 = 14450$, etc.

6. Let x be the greater ; then $\frac{377}{x}$ is the less ; and $x - 21 = 21 - \frac{377}{x}$; $x^2 - 42x = -377$; $x^2 - 42x + 441 = 64$, etc.

7. Let x be the number ; then $\frac{x}{2} \times \frac{x}{3} \times \frac{x}{4} \times \frac{x}{5} = 6750$; $x^4 = 810000$; $x^2 = 900$; $x = 30$.

8. Let x be the number ; then $\frac{11500}{x} = x + \frac{51}{x}$; $11500 = x^2 + 51$, etc.

9. Let x be the number ; then $(x + 20)^2 + 2\,(x - 10)^2 = 17475$; $x^2 + 40x + 400 + 2x^2 - 40x + 200 = 17475$; $3x^2 = 16875$, etc.

10. Let x and $26 - x$ be the numbers ; then $x^2 + (26 - x)^2 = 436$; $2x^2 - 52x + 676 = 436$; $x^2 - 26x = -120$, etc.

11. Let $x + 17$ and x be the numbers ; then $(x + 17)^2 + x^2 = 325$, etc.

12. Let x and $\frac{255}{x}$ be the numbers ; then $x^2 + \frac{65025}{x^2} = 514$;
$x^4 - 514x^2 = -65025$; $x^4 - 514x^2 + 66049 = 1024$;
$x^2 - 257 = \pm 32$; $x^2 = 289$ or 225 ; $x = 17$ or 15.

13. Let x and $16 - x$ be the parts ; then $x^2 + (16 - x)^2 + (16 - x)x = 208$
$x^2 - 16x + 256 = 208$; $x^2 - 16x = -48$, etc.

14. Let x^2 be the number ; then $x^2 + x = 1332$; $x^2 + x + \frac{1}{4} = \frac{532}{4}$
$x + \frac{1}{2} = \pm \frac{73}{2}$; $x = 36$; $x^2 = 1296$.

15. Let x^2 be the number; then $x^2-x=\frac{195}{4}$, etc.

16. Let x^2 be the number; then $x^2-x=2550$, etc.

17. Let x and $\frac{24}{x}$ be the numbers; then $\left(x+\frac{24}{x}\right)\left(x-\frac{24}{x}\right)=20$;

$x^2-\frac{576}{x^2}=20$; $x^4-20x^2=576$; $x^4-20x^2+100=676$;

$x^2-10=\pm 26$; taking the positive value $x=6$, etc.

18. Let x be the greater, y the less; then $x\,(x+y)=204$ and

$y\,(x-y)=35$, or, $x^2+xy=204$ and $xy-y^2=35$.

Put $y=mx$. Then $\frac{x^2+mx^2}{mx^2-m^2x^2}=\frac{204}{35}$; hence one value of m is $\frac{5}{12}$

and $x^2+\frac{5x^2}{12}=204$; $17x^2=2448$; $x^2=144$, etc.

19. Let x and $x-5$ be the numbers; then $x\,(2x-5)=228$, etc.

20. Let $x-1$, x, $x+1$ be the numbers; then $(x-1)\,x\,(x+1)=3x$;

$x^2-1=3$; $x^2=4$; $x=2$, etc.

21. Let x and $x+1$ be the numbers; then $(x+1)^2-x^2=15$, etc.

22. Let x and $x+1$ be the numbers; then $x^2+(x+1)^2=481$, etc.

23. Let $x-1$, x and $x+1$ be the numbers; then $(x-1)^2+x^2+(x+1)^2$

$=365$, etc.

24. Let x be the number; then $\frac{60}{x}=\frac{60}{x+3}+1$;

$60x+180=60x+x^2+3x$; $x^2+3x=180$, etc.

25. Let x be the number; then $\frac{80}{x}=\frac{80}{x+4}+1$, etc.

26. Let x be the price of each piece in shillings; then $675 = 48 \times \frac{675}{x} - x$;
$675x = 32400 - x^2$, etc.

27. Let x be the number; then $\frac{180}{x} = \frac{180}{x+3} + 3$, etc.

28. Let x be the number of miles; then $\frac{108}{x} = \frac{108}{x+3} + 6$, etc.

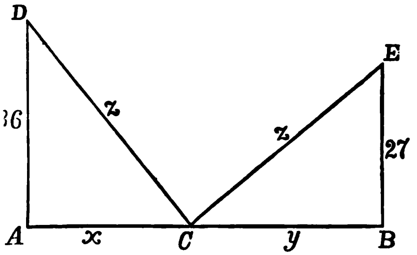
29. Let x be the number of sheep; then $\frac{60}{x} = \frac{54}{x-15} - \frac{1}{10}$, etc.

30. Let x and $x+2$ be the number of hours; then $\frac{1}{x} + \frac{1}{x+2} = \frac{1}{2\frac{11}{12}}$, etc.

31. Let x be the number of yards in the breadth;
then $x(x+1) = 10100$, etc.

32. Let $10x + y$ be the number; then $x = 2y$ and $(10x+y)(10y+x)$
$= 2268$; hence $21y \times 12y = 2268$; $y^2 = 9$, etc.

33. Let x and y be the distances in feet of the ladder from the bottoms of the houses; z the length of the ladder in feet.



Then since DCE is a right angle,

DCA and ECB together make a right angle. Eucl. i. 13.

But DCA and ADC together make a right angle. Eucl. i. 32.

$\therefore$ angle $ECB =$ angle ADC.

Hence triangles ACD and BEC are equal in all respects. Eucl. i. 26.

$\therefore$ $x = 27$, and $y = 36$; and $x + y = 63$.

Also $z^2 = (27)^2 + (36)^2$. Eucl. i. 47; and $\therefore$ $z = 45$.

34. Let x and y be the length and width in feet.

$$\left.\begin{aligned}&\text{Then } \frac{7x}{8}\times\frac{15y}{16}=xy-\frac{23}{4}\\&\text{and } 2\left(\frac{7x}{8}+\frac{15y}{16}\right)=2(x+y)-\frac{17}{4}\end{aligned}\right\};$$

$$\left.\begin{aligned}105xy&=128xy-736\\14x+15y&=16x+16y-34\end{aligned}\right\};\quad\left.\begin{aligned}xy&=32\\2x+y&=34\end{aligned}\right\};\ 34x-2x^2=32,\text{ etc.}$$

35. Let $10x+y$ be the number. Since the number is less than 50, and the difference of the digits 4, y is the greater of the digits.

$$\left.\begin{aligned}&\text{Then } y-x=4\\&(10y+x)^2-(10x+y)^2=3960\end{aligned}\right\},\text{ etc.}$$

36. Let x be the number of rows, $\frac{10000}{x}$ is then the number of trees in each row.

Then $(x-20)\left(\frac{10000}{x}+25\right)=10000$

or, $10000x-200000+25x^2-500x=10000x$

or, $x^2-20x=8000$, etc.

37. Let x be the number of men in the regiment; y the number in the side of the square at the first attempt.

Then $y^2=x-39$ and $(y+1)^2=x+50$.

Hence $(y+1)^2-y^2=89$; whence $2y=88$, $y=44$, etc.

XCIX.

1. $x=\frac{29-7y}{5}=5-y+\frac{4-2y}{5}$; let $\frac{4-2y}{5}=m$

$y=\frac{4-5m}{2}=2-2m-\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=2-4n-n=2-5n$

and $x=5-y+m=5-2+5n+2n=3+7n$

when $n=0$; $y=2$ and $x=3$.

2. $x=\dfrac{92-19y}{7}=13-2y-\dfrac{5y-1}{7}$; let $\dfrac{5y-1}{7}=m$

$y=\dfrac{7m+1}{5}=m+\dfrac{2m+1}{5}$; let $\dfrac{2m+1}{5}=n$

$m=\dfrac{5n-1}{2}=2n+\dfrac{n-1}{2}$; let $\dfrac{n-1}{2}=p$

$n=2p+1$

Then $y=m+n=2n+p+n=4p+2+p+2p+1=7p+3$

$x=13-14p-6-4p-2-p=5-19p$

when $p=0$, $x=5$ and $y=3$.

3. $x=\dfrac{1170-19y}{13}=90-y-\dfrac{6y}{13}$; let $\dfrac{6y}{13}=m$

$y=\dfrac{13m}{6}=2m+\dfrac{m}{6}$; let $\dfrac{m}{6}=n$; $m=6n$

Then $y=13n$ and $x=90-19n$

when $n=0$, $y=0$ and $x=90$

$n=1$, $y=13$ and $x=71$, etc.

4. $x=\dfrac{26-5y}{3}=8-y-\dfrac{2y-2}{3}$; let $\dfrac{2y-2}{3}=m$

$y=\dfrac{3m+2}{2}=m+1+\dfrac{m}{2}$; let $\dfrac{m}{2}=n$; $m=2n$

Then $y=2n+1+n=3n+1$

$x=8-3n-1-2n=7-5n$

when $n=0$, $y=1$, $x=7$

$n=1$, $y=4$, $x=2$.

5. $5y=14x-7$; $y=2x-1+\dfrac{4x-2}{5}$; let $\dfrac{4x-2}{5}=m$

$x=\dfrac{5m+2}{4}=m+\dfrac{m+2}{4}$; let $\dfrac{m+2}{4}=n$; $m=4n-2$

Then $x=4n-2+n=5n-2$

$y=10n-4-1+4n-2=14n-7$, etc.

6. $x=\frac{1031-15y}{11}=93-y-\frac{4y-8}{11}$; let $\frac{4y-8}{11}=m$

$y=\frac{11m+8}{4}=2m+2+\frac{3m}{4}$; let $\frac{3m}{4}=n$

$m=\frac{4n}{3}=n+\frac{n}{3}$; let $\frac{n}{3}=p$; $n=3p$

Then $y=8p+2+3p=11p+2$

$x=93-11p-2-4p=91-15p$, etc.

7. $y=\frac{308-11x}{7}=44-x-\frac{4x}{7}$; let $\frac{4x}{7}=m$

$x=\frac{7m}{4}=m+\frac{3m}{4}$; let $\frac{3m}{4}=n$

$m=\frac{4n}{3}=n+\frac{n}{3}$; let $\frac{n}{3}=p$; $n=3p$

Then $x=m+n=3p+p+3p=7p$

$y=44-x-m=44-7p-4p=44-11p$, etc.

8. $x=\frac{23+19y}{4}=5+4y+\frac{3y+3}{4}$; let $\frac{3y+3}{4}=m$

$y=\frac{4m-3}{3}=m-1+\frac{m}{3}$; let $\frac{m}{3}=n$; $m=3n$

Then $y=3n-1+n=4n-1$

$x=5+16n-4+3n=19n+1$, etc.

9. $y=\frac{20x-683}{9}=2x-75+\frac{2x-8}{9}$; let $\frac{2x-8}{9}=m$

$x=\frac{9m+8}{2}=4m+4+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $x=8n+4+n=9n+4$

$y=18n+8-75+2n=20n-67$, etc.

10. $x=\frac{383-7y}{3}=127-2y-\frac{y-2}{3}$; let $\frac{y-2}{3}=m$

Then $y=3m+2$

$x=127-6m-4-m=123-7m$. etc.

11. $y=\frac{54-27x}{4}=13-6x-\frac{3x-2}{4}$; let $\frac{3x-2}{4}=m$

$x=\frac{4m+2}{3}=m+\frac{m+2}{3}$; let $\frac{m+2}{3}=n$; $m=3n-2$

Then $x=3n-2+n=4n-2$

$y=13-24n+12-3n+2=27-27n$, etc.

12. $x=\frac{653-9y}{7}=93-y-\frac{2y-2}{7}$; let $\frac{2y-2}{7}=m$

$y=\frac{7m+2}{2}=3m+1+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $y=6n+1+n=7n+1$

$x=93-7n-1-2n=92-9n$, etc.

13. $\frac{x}{7}+\frac{y}{9}=\frac{57}{63}$; $9x+7y=57$

$y=\frac{57-9x}{7}=8-x-\frac{2x-1}{7}$; let $\frac{2x-1}{7}=m$

$x=\frac{7m+1}{2}=3m+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=6n-3+n=7n-3$

$y=8-7n+3-2n+1=12-9n$

when $n=1$, $x=4$, $y=3$.

14. $\frac{x}{11}-\frac{y}{13}=\frac{82}{143}$; $13x-11y=82$

$y=\frac{13x-82}{11}=x-7+\frac{2x-5}{11}$; let $\frac{2x-5}{11}=m$

$x=\frac{11m+5}{2}=5m+2+\frac{m+1}{2}$; let $\frac{m+1}{2}=n$; $m=2n-1$

Then $x=10n-5+2+n=11n-3$

$y=11n-3-7+2n-1=13n-11$

when $n=1$, $x=8$, $y=2$.

15. Let x be the number of florins, y the number of half-crowns; then

$4x+5y=58$; $x=14-y-\frac{y-2}{4}$; let $\frac{y-2}{4}=m$

Then $y=4m+2$ and $x=12-5m$, etc.

16. Let x be the number of half-guineas, y the number of half-crowns; then

$21x+5y=800$; $y=160-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$ and $y=160-21m$

when $m=1$, $x=5$, $y=139$, etc.

17. Let x be the number; y the first quotient, z the second quotient.

Then $\frac{x}{5}=y+\frac{2}{5}$; $\frac{x}{9}=z+\frac{3}{9}$

Hence $5y+2=9z+3$; $y=z+\frac{4z+1}{5}$; let $\frac{4z+1}{5}=m$

$z=\frac{5m-1}{4}=m+\frac{m-1}{4}$; let $\frac{m-1}{4}=n$; $m=4n+1$

$z=5n+1$; $y=9n+2$

Hence $y=2, 11 \ldots$ and hence $x=12, 57 \ldots$

18. Let x be the number of guineas, and y the number of crowns.

Then $21x+5y=235$; $y=47-4x-\frac{x}{5}$; let $\frac{x}{5}=m$

Then $x=5m$; and $y=47-21m$, etc.

19. Let x be the number of half-guineas, and y the number of half-crowns.

Then $21x+5y=183$; $y=36-4x-\frac{x-3}{5}$ let $\frac{x-3}{5}=m$

Then $x=5m+3$; and $y=36-20m-12-m=24-21m$, etc.

20. Divide by 17; then $19x-31y=\frac{1000}{17}$; and the right-hand side being a fraction, no integral values of x and y can make the left side equal to the right.

21. Let x be the number of oxen, y of sheep, z of hens.

Then $x+y+z=100$, and $100x+20y+z=2000$,

$\therefore 99x+19y=1900$

Hence $y=100-5x-\frac{4x}{19}$; let $\frac{4x}{19}=m$

$x=\frac{19m}{4}=5m-\frac{m}{4}$; let $\frac{m}{4}=n$; $m=4n$

Then $x=19n$; $y=100-99n$

If $n=1$, $x=19$, $y=1$, and $\therefore z=80$.

22. Let A give x sixpences and receive y fourpenny pieces.

Then $6x-4y=58$; or $3x-2y=29$

Then $y=x-14+\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

Then $x=2m+1$ and $y=3m-13$, etc.

23. Let x be the number of half-crowns, y of florins, z of shillings.

Then $x+y=4z$, and $\frac{5x}{2}+2y+z=244$

Hence $x+y=4z$, and $10x+8y+4z=976$

$\therefore 11x+9y=976$

Hence $y=108-x-\frac{2x-4}{9}$; let $\frac{2x-4}{9}=m$

$x=4m+2+\frac{m}{2}$; let $\frac{m}{2}=n$; $m=2n$

Then $x=9n+2$; $y=108-9n-2-2n=106-11n$

Hence $x=2$, $y=106$, $z=27$, which gives the greatest number of coins possible.

24. Let there be x half-crowns, y florins, and z fourpenny-pieces.

Then $x+y+z=50$, and $30x+24y+4z=1200$.

Hence $26x+20y=1000$.

$\therefore y=50-x-\frac{3x}{10}$; let $\frac{3x}{10}=m$.

Then $x=\frac{10m}{3}=3m+\frac{m}{3}$; let $\frac{m}{3}=n$; $m=3n$.

Hence $x=10n$ and $y=50-13n$

$\therefore x=10, 20, 30$; $y=37, 24, 11$; $z=3, 6, 9$.

25. Let A give x sovereigns, and receive y dollars.

Then $240x-51y=12$; or, $80x-17y=4$.

Hence $y=4x+\frac{12x-4}{17}$; let $\frac{12x-4}{17}=m$

$x=m+\frac{5m+4}{12}$; let $\frac{5m+4}{12}=n$

$m=2n+\frac{2n-4}{5}$; let $\frac{2n-4}{5}=p$

$n=2p+2+\frac{p}{2}$; let $\frac{p}{2}=q$; $p=2q$.

Then $x=m+n=3n+p=6p+6+5q=17q+6$

$y=4x+m=68q+24+2n+p=82q+28$.

Hence when $q=0$, $x=6$, $y=28$.

26. Let $2x$ and $3y$ be the parts; then $2x+3y=25$.

Then $x=12-y-\frac{y-1}{2}$; let $\frac{y-1}{2}=m$

$y=2m+1$ and $x=11-3m$

When $m=0$, $y=1$, $x=11$; $2x=22$, $3y=3$, etc.

27. Let there be x crowns and y florins; then $5x+2y=49$.

Then $y=24-2x-\frac{x-1}{2}$; let $\frac{x-1}{2}=m$

$x=2m+1$, and $y=22-5m$

Hence we get positive values for x and y, when $m=0, 1, 2, 3, 4$.

28. Let $7x$ and $11y$ be the parts; then $7x+11y=100$.

Then $x=14-y-\dfrac{4y-2}{7}$; let $\dfrac{4y-2}{7}=m$.

$$y=\frac{7m+2}{4}=m+\frac{3m+2}{4}\text{; let }\frac{3m+2}{4}=n$$

$$m=\frac{4n-2}{3}=n+\frac{n-2}{3}\text{; let }\frac{n-2}{3}=p\text{; }n=3p+2.$$

Hence $y=2n+p=7p+4$

$$x=14-7p-4-n-p=8-11p.$$

When $p=0$, $y=4$, $x=8$; $7x=56$, $11y=44$.

29. Let the quotients be x and y.

Then $5x+2$ and $7y+4$ are the dividends

$\therefore 5x+7y+6=100$, or, $5x+7y=94$.

Then $x=18-y-\dfrac{2y-4}{5}$; let $\dfrac{2y-4}{5}=m$

$$y=2m+2+\frac{m}{2}\text{; let }\frac{m}{2}=n\text{; }m=2n.$$

Then $y=5n+2$ and $x=16-7n$.

Hence $x=16, 9, 2$; and $y=2, 7, 12$

and $5x+2=82, 47, 12$; and $7y+4=18, 53, 88$.

30. The L. C. M. of 2, 3, 4, 5, 6 is 60.

Hence the general expression for the number required is $60m+1$.

Making $m=1, 2, 3$ . . . . successively, we get 61, 121, 181, 241, 301, 361, and of these 301 is the only multiple of 7.

CI.

1.

$$\begin{array}{lllll}
x^{2p}+x^{p}y^{p}+y^{2p} & & & & \\
x^{2p}-x^{p}y^{p}+y^{2p} & & & & \\
\hline
x^{4p} & +x^{3p}y^{p} & +x^{2p}y^{2p} & & \\
 & -x^{3p}y^{p} & -x^{2p}y^{2p} & -x^{p}y^{3p} & \\
 & & +x^{2p}y^{2p} & +x^{p}y^{3p} & +y^{4p} \\
\hline
x^{4p} & & +x^{2p}y^{2p} & & +y^{4p}
\end{array}$$

2.
$$\begin{array}{l}
a^{3m}+3a^{2m}y^{n}+9a^{m}y^{2n}+27y^{3n}\\
a^{m}-3y^{n}\\
\hline
a^{4m}+3a^{3m}y^{n}+9a^{2m}y^{2n}+27a^{m}y^{3n}\\
\qquad -3a^{3m}y^{n}-9a^{2m}y^{2n}-27a^{m}y^{3n}-81y^{4n}\\
\hline
a^{4m} \qquad\qquad\qquad\qquad\qquad\qquad\qquad -81y^{4n}
\end{array}$$

3.
$$\begin{array}{l}
x^{4d}-2ax^{2d}+4a^{2}\\
x^{4d}+2ax^{2d}+4a^{2}\\
\hline
x^{8d}-2ax^{6d}+4a^{2}x^{4d}\\
\qquad +2ax^{6d}-4a^{2}x^{4d}+8a^{3}x^{2d}\\
\qquad\qquad\qquad +4a^{2}x^{4d}-8a^{3}x^{2d}+16a^{4}\\
\hline
x^{8d} \qquad\qquad +4a^{2}x^{4d} \qquad\qquad +16a^{4}
\end{array}$$

4.
$$\begin{array}{l}
a^{m}+b^{n}+c^{r}\\
a^{m}-b^{n}+c^{r}\\
\hline
a^{2m}+a^{m}b^{n}+a^{m}c^{r}\\
\qquad -a^{m}b^{n}-b^{2n}-b^{n}c^{r}\\
\qquad\qquad\qquad +a^{m}c^{r}+b^{n}c^{r}+c^{2r}\\
\hline
a^{2m}-b^{2n}+c^{2r}+2a^{m}c^{r}
\end{array}$$

5.
$$\begin{array}{l}
a^{m}+b^{n}-2c^{r}\\
2a^{m}-b+c^{2}\\
\hline
2a^{2m}+2a^{m}b^{n}-4a^{m}c^{r}\\
\qquad -a^{m}b-b^{n+1}+2bc^{r}\\
\qquad\qquad +a^{m}c^{2}+b^{n}c^{2}-2c^{r+2}\\
\hline
2a^{2m}+2a^{m}b^{n}-4a^{m}c^{r}-a^{m}b-b^{n+1}+2bc^{r}+a^{m}c^{2}+b^{n}c^{2}-2c^{r+2}
\end{array}$$

6.
$$\begin{array}{l}
x^{mn-n}-y^{m}\\
x^{n}+y^{mn-n}\\
\hline
x^{mn}-x^{n}y^{m}\\
\qquad +x^{mn-n}.y^{mn-n}-y^{mn-n+m}\\
\hline
x^{mn}+x^{mn-n}.y^{mn-n}-x^{n}y^{m}-y^{mn-n+m}
\end{array}$$

7.
$$\begin{array}{l}
x^{2n}-x^ny^n+y^{2n} \\
x^{2n}+x^ny^n+y^{2n} \\
\hline
x^{4n}-x^{3n}y^n+x^{2n}y^{2n} \\
\qquad +x^{3n}y^n-x^{2n}y^{2n}+x^ny^{3n} \\
\qquad\qquad +x^{2n}y^{2n}-x^ny^{3n}+y^{4n} \\
\hline
x^{4n}\qquad +x^{2n}y^{2n}\qquad +y^{4n}
\end{array}$$

8.
$$\begin{array}{l}
a^{p^2+p}-b^{p^2}+c^p \\
a^{p^2-p}+b^{1-p^2}+c^{1-p} \\
\hline
a^{2p^2}-a^{p^2-p}b^{p^2}+a^{p^2-p}c^p \\
\qquad +a^{p^2+p}b^{1-p^2}-b+b^{1-p^2}c^p \\
\qquad\qquad\qquad +a^{p^2+p}c^{1-p}-b^{p^2}c^{1-p}+c \\
\hline
a^{2p^2}-a^{p^2-p}b^{p^2}+a^{p^2-p}c^p+a^{p^2+p}b^{1-p^2}-b+b^{1-p^2}c^p \\
\qquad\qquad\qquad\qquad +a^{p^2+p}c^{1-p}-b^{p^2}c^{1-p}+c
\end{array}$$

9.
$$\begin{array}{l}
x^{2p}+x^p+1 \\
x^{2p}+x^p+1 \\
\hline
x^{4p}+x^{3p}+x^{2p} \\
\quad +x^{3p}+x^{2p}+x^p \\
\qquad +x^{2p}+x^p+1 \\
\hline
x^{4p}+2x^{3p}+3x^{2p}+2x^p+1
\end{array}$$

10.
$$\begin{array}{l}
x^{2p}-x^p+1 \\
x^{2p}-x^p+1 \\
\hline
x^{4p}-x^{3p}+x^{2p} \\
\quad -x^{3p}+x^{2p}-x^p \\
\qquad +x^{2p}-x^p+1 \\
\hline
x^{4p}-2x^{3p}+3x^{2p}-2x^p+1
\end{array}$$

CII.

1.
$$\begin{array}{l}
x^m-y^m)\,x^{4m}-y^{4m}\,(x^{3m}+x^{2m}y^m+x^my^{2m}+y^{3m} \\
\qquad \underline{x^{4m}-x^{3m}y^m} \\
\qquad\qquad x^{3m}y^m-y^{4m} \\
\qquad\qquad \underline{x^{3m}y^m-x^{2m}y^{2m}} \\
\qquad\qquad\qquad x^{2m}y^{2m}-y^{4m} \\
\qquad\qquad\qquad \underline{x^{2m}y^{2m}-x^my^{3m}} \\
\qquad\qquad\qquad\qquad x^my^{3m}-y^{4m} \\
\qquad\qquad\qquad\qquad x^my^{3m}-y^{4m}
\end{array}$$

2.
$$\begin{array}{l}
x^n+y^n)\,x^{5n}+y^{5n}\,(x^{4n}-x^{3n}y^n+x^{2n}y^{2n}-x^ny^{3n}+y^{4n}\\
\qquad\qquad\underline{x^{5n}+x^{4n}y^n}\\
\qquad\qquad\qquad -x^{4n}y^n+y^{5n}\\
\qquad\qquad\qquad \underline{-x^{4n}y^n-x^{3n}y^{2n}}\\
\qquad\qquad\qquad\qquad\qquad x^{3n}y^{2n}+y^{5n}\\
\qquad\qquad\qquad\qquad\qquad \underline{x^{3n}y^{2n}+x^{2n}y^{3n}}\\
\qquad\qquad\qquad\qquad\qquad\qquad -x^{2n}y^{3n}+y^{5n}\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{-x^{2n}y^{3n}-x^ny^{4n}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^ny^{4n}+y^{5n}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^ny^{4n}+y^{5n}}
\end{array}$$

3.
$$\begin{array}{l}
x^r-y^r)\,x^{6r}-y^{6r}\,(x^{5r}+x^{4r}y^r,\text{ etc.}\\
\qquad\quad\underline{x^{6r}-x^{5r}y^r}\\
\qquad\qquad\quad x^{5r}y^r-y^{6r},\text{ etc.}
\end{array}$$

4.
$$\begin{array}{l}
a^{3p}+b^{2q})\,a^{15p}+b^{10q}\,(a^{12p}-a^{9p}b^{2q},\text{ etc.}\\
\qquad\qquad\underline{a^{15p}+a^{12p}b^{2q}}\\
\qquad\qquad\qquad -a^{12p}b^{2q}+b^{10q},\text{ etc.}
\end{array}$$

5.
$$\begin{array}{l}
x^d-3)\,x^{5d}-243)\,x^{4d}+3x^{3d},\text{ etc.}\\
\qquad\quad\underline{x^{5d}-3x^{4d}}\\
\qquad\qquad\quad 3x^{4d}-243,\text{ etc.}
\end{array}$$

6.
$$\begin{array}{l}
a^{2m}+2a^mx^n+4x^{2n})\,a^{4m}+4a^{2m}x^{2n}+16x^{4n}\,(a^{2m}-2a^mx^n+4x^{2n}\\
\qquad\qquad\qquad\quad\underline{a^{4m}+2a^{3m}x^n+4a^{2m}x^{2n}}\\
\qquad\qquad\qquad\qquad -2a^{3m}x^n+16x^{4n}\\
\qquad\qquad\qquad\qquad \underline{-2a^{3m}x^n-4a^{2m}x^{2n}-8a^mx^{3n}}\\
\qquad\qquad\qquad\qquad\qquad\quad 4a^{2m}x^{2n}+8a^mx^{3n}+16x^{4n}\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{4a^{2m}x^{2n}+8a^mx^{3n}+16x^{4n}}
\end{array}$$

7. $1 + 5x^p + x^{2p})\,2 + 9x^p + 14x^{3p} + 3x^{4p}\,(2 - x^p + 3x^{2p}$

$2 + 10x^p + 2x^{2p}$

$-x^p - 2x^{2p} + 14x^{3p}$

$-x^p - 5x^{2p} - x^{3p}$

$3x^{2p} + 15x^{3p} + 3x^{4p}$

$3x^{2p} + 15x^{3p} + 3x^{4p}$

8. $b^mc^{2m} - 2b^{2m}c^m + b^{3m})\,4b^{2m}c^{3m} - 13b^{3m}c^{2m} + 14b^{4m}c^m - 5b^{5m}\,(4b^mc^m - 5b^{2m}$

$4b^{2m}c^{3m} - 8b^{3m}c^{2m} + 4b^{4m}c^m$

$-5b^{3m}c^{2m} + 10b^{4m}c^m - 5b^{5m}$

$-5b^{3m}c^{2m} + 10b^{4m}c^m - 5b^{5m}$

9. $a^{6m} + 6a^{5m} + 15a^{4m} + 20a^{3m} + 15a^{2m} + 6a^m + 1\,(a^{3m} + 3a^{2m} + 3a^m + 1$

a^{6m}

$2a^{3m} + 3a^{2m}$ \ $6a^{5m} + 15a^{4m}$

$6a^{5m} + 9a^{4m}$

$2a^{3m} + 6a^{2m} + 3a^m$ \ $6a^{4m} + 20a^{3m} + 15a^{2m}$

$6a^{4m} + 18a^{3m} + 9a^{2m}$

$2a^{3m} + 6a^{2m} + 6a^m + 1$ \ $2a^{3m} + 6a^{2m} + 6a^m + 1$

$2a^{3m} + 6a^{2m} + 6a^m + 1$

10. $a^{2m} + 2a^mb^n + b^{2n} + 2a^mc^r + 2b^nc^r + c^{2r}\,(a^m + b^n + c^r$

a^{2m}

$2a^m + b^n$ \ $2a^mb^n + b^{2n}$

$2a^mb^n + b^{2n}$

$2a^m + 2b^n + c^r$ \ $2a^mc^r + 2b^nc^r + c^{2r}$

$2a^mc^r + 2b^nc^r + c^{2r}$

CIII.

1.
$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$
$x^{\frac{1}{3}} - 1$
$x - 2x^{\frac{2}{3}} + x^{\frac{1}{3}}$
$\quad - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 1$
$x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1$

2.
$y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}} + 1$
$y^{\frac{1}{4}} - 1$
$y + y^{\frac{3}{4}} + y^{\frac{1}{2}} + y^{\frac{1}{4}}$
$\quad - y^{\frac{3}{4}} - y^{\frac{1}{2}} - y^{\frac{1}{4}} - 1$
$y \qquad\qquad - 1$

3.
$a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$
$a^{\frac{2}{3}} - x^{\frac{2}{3}}$
$a^2 + a^{\frac{4}{3}}x^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{4}{3}}$
$\quad - a^{\frac{4}{3}}x^{\frac{2}{3}} - a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2$
$a^2 \qquad\qquad - x^2$

4.
$a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}}$
$a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$
$a - a^{\frac{2}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{2}{3}} - a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + a^{\frac{1}{3}}c^{\frac{2}{3}}$
$\quad + a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{2}{3}}c^{\frac{1}{3}} + b^{\frac{1}{3}}c^{\frac{2}{3}}$
$\qquad + a^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}} + b^{\frac{2}{3}}c^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{2}{3}} + c$
$a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$

5.
$5x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{4}} + 3x^{\frac{1}{3}}y^{\frac{1}{2}} + 7y^{\frac{3}{4}}$
$2x^{\frac{1}{3}} - 3y^{\frac{1}{4}}$
$10x + 4x^{\frac{2}{3}}y^{\frac{1}{4}} + 6x^{\frac{1}{3}}y^{\frac{1}{2}} + 14x^{\frac{1}{3}}y^{\frac{3}{4}}$
$\quad - 15x^{\frac{2}{3}}y^{\frac{1}{4}} - 6x^{\frac{1}{3}}y^{\frac{1}{2}} - 9x^{\frac{1}{3}}y^{\frac{3}{4}} - 21y$
$10x - 11x^{\frac{2}{3}}y^{\frac{1}{4}} \qquad + 5x^{\frac{1}{3}}y^{\frac{3}{4}} - 21y$

6.
$m^{\frac{4}{5}} + m^{\frac{3}{5}}n^{\frac{1}{5}} + m^{\frac{2}{5}}n^{\frac{2}{5}} + m^{\frac{1}{5}}n^{\frac{3}{5}} + n^{\frac{4}{5}}$
$m^{\frac{1}{5}} - n^{\frac{1}{5}}$
$m + m^{\frac{4}{5}}n^{\frac{1}{5}} + m^{\frac{3}{5}}n^{\frac{2}{5}} + m^{\frac{2}{5}}n^{\frac{3}{5}} + m^{\frac{1}{5}}n^{\frac{4}{5}}$
$\quad - m^{\frac{4}{5}}n^{\frac{1}{5}} - m^{\frac{3}{5}}n^{\frac{2}{5}} - m^{\frac{2}{5}}n^{\frac{3}{5}} - m^{\frac{1}{5}}n^{\frac{4}{5}} - n$
$m \qquad\qquad - n$

7.

$$m^{\frac23} - 2d^{\frac14}m^{\frac13} + 4d^{\frac12}$$
$$m^{\frac23} + 2d^{\frac14}m^{\frac13} + 4d^{\frac12}$$
$$m^{\frac43} - 2d^{\frac14}m \quad + 4d^{\frac12}m^{\frac23}$$
$$+ 2d^{\frac14}m \quad - 4d^{\frac12}m^{\frac23} + 8d^{\frac34}m^{\frac13}$$
$$+ 4d^{\frac12}m^{\frac23} - 8d^{\frac34}m^{\frac13} + 16d$$
$$m^{\frac43} \qquad + 4d^{\frac12}m^{\frac23} \qquad + 16d$$

8.

$$8a^{\frac34} + 4a^{\frac12}b^{\frac14} + 5a^{\frac14}b^{\frac12} + 9b^{\frac34}$$
$$2a^{\frac14} - 3b^{\frac14}$$
$$16a + 8a^{\frac34}b^{\frac14} + 10a^{\frac12}b^{\frac12} + 18a^{\frac14}b^{\frac34}$$
$$- 24a^{\frac34}b^{\frac14} - 12a^{\frac12}b^{\frac12} - 15a^{\frac14}b^{\frac34} - 27b$$
$$16a + 8a^{\frac34}b^{\frac14} + 10a^{\frac12}b^{\frac12} + 18a^{\frac14}b^{\frac34} - 24a^{\frac34}b^{\frac14}, \text{ etc.}$$

9.

$$x^{\frac13} + a^{\frac13}$$
$$x^{\frac13} + a^{\frac13}$$
$$x^{\frac23} + a^{\frac13}x^{\frac13}$$
$$+ a^{\frac13}x^{\frac13} + a^{\frac23}$$
$$x^{\frac23} + 2a^{\frac13}x^{\frac13} + a^{\frac23}$$

10.

$$x^{\frac13} - a^{\frac13}$$
$$x^{\frac13} - a^{\frac13}$$
$$x^{\frac23} - a^{\frac13}x^{\frac13}$$
$$- a^{\frac13}x^{\frac13} + a^{\frac23}$$
$$x^{\frac23} - 2a^{\frac13}x^{\frac13} + a^{\frac23}$$

11.

$$x^{\frac23} + y^{\frac23}$$
$$x^{\frac23} + y^{\frac23}$$
$$x^{\frac43} + x^{\frac23}y^{\frac23}$$
$$+ x^{\frac23}y^{\frac23} + y^{\frac43}$$
$$x^{\frac43} + 2x^{\frac23}y^{\frac23} + y^{\frac43}$$

12.

$$a + b^{\frac13}$$
$$a + b^{\frac13}$$
$$a^2 + ab^{\frac13}$$
$$+ ab^{\frac13} + b^{\frac23}$$
$$a^2 + 2ab^{\frac13} + b^{\frac23}$$

13.

$$x^{\frac12} - 2x^{\frac14} + 3$$
$$x^{\frac12} - 2x^{\frac14} + 3$$
$$x - 2x^{\frac34} + 3x^{\frac12}$$
$$- 2x^{\frac34} + 4x^{\frac12} - 6x^{\frac14}$$
$$+ 3x^{\frac12} - 6x^{\frac14} + 9$$
$$x - 4x^{\frac34} + 10x^{\frac12} - 12x^{\frac14} + 9$$

14.

$$2x^{\frac25} + 3x^{\frac15} + 4$$
$$2x^{\frac25} + 3x^{\frac15} + 4$$
$$4x^{\frac45} + 6x^{\frac35} + 8x^{\frac25}$$
$$+ 6x^{\frac35} + 9x^{\frac25} + 12x^{\frac15}$$
$$+ 8x^{\frac25} + 12x^{\frac15} + 16$$
$$4x^{\frac45} + 12x^{\frac35} + 25x^{\frac25} + 24x^{\frac15} + 16$$

15.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}\\
\underline{x^{\frac{1}{3}} - y^{\frac{1}{3}} + z^{\frac{1}{3}}}\\
x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{1}{3}}\\
\quad - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}}\\
\qquad\qquad + x^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}\\
\hline
x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}}z^{\frac{1}{3}} - 2y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}}
\end{array}$$

16.
$$\begin{array}{l}
x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}}\\
\underline{x^{\frac{1}{4}} + 2y^{\frac{1}{4}} - z^{\frac{1}{4}}}\\
x^{\frac{1}{2}} + 2x^{\frac{1}{4}}y^{\frac{1}{4}} - x^{\frac{1}{4}}z^{\frac{1}{4}}\\
\quad + 2x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}}\\
\qquad\qquad - x^{\frac{1}{4}}z^{\frac{1}{4}} - 2y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}\\
\hline
x^{\frac{1}{2}} + 4x^{\frac{1}{4}}y^{\frac{1}{4}} + 4y^{\frac{1}{2}} - 2x^{\frac{1}{4}}z^{\frac{1}{4}} - 4y^{\frac{1}{4}}z^{\frac{1}{4}} + z^{\frac{1}{2}}
\end{array}$$

CIV.

1.
$$\begin{array}{l}
x^{\frac{1}{2}} - y^{\frac{1}{2}})\, x - y\, (x^{\frac{1}{2}} + y^{\frac{1}{2}}\\
\qquad\qquad \underline{x - x^{\frac{1}{2}}y^{\frac{1}{2}}}\\
\qquad\qquad\quad x^{\frac{1}{2}}y^{\frac{1}{2}} - y\\
\qquad\qquad\quad \underline{x^{\frac{1}{2}}y^{\frac{1}{2}} - y}
\end{array}$$

2.
$$\begin{array}{l}
a^{\frac{1}{2}} + b^{\frac{1}{2}})\, a - b\, (a^{\frac{1}{2}} - b^{\frac{1}{2}}\\
\qquad\qquad \underline{a + a^{\frac{1}{2}}b^{\frac{1}{2}}}\\
\qquad\qquad\quad - a^{\frac{1}{2}}b^{\frac{1}{2}} - b\\
\qquad\qquad\quad \underline{- a^{\frac{1}{2}}b^{\frac{1}{2}} - b}
\end{array}$$

3.
$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}})\, x - y\, (x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}\\
\qquad\qquad \underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}}\\
\qquad\qquad\quad x^{\frac{2}{3}}y^{\frac{1}{3}} - y\\
\qquad\qquad\quad \underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}}}\\
\qquad\qquad\qquad\quad x^{\frac{1}{3}}y^{\frac{2}{3}} - y\\
\qquad\qquad\qquad\quad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

4.
$$\begin{array}{l}
a^{\frac{1}{3}} + b^{\frac{1}{3}})\, a + b\, (a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\\
\qquad\qquad \underline{a + a^{\frac{2}{3}}b^{\frac{1}{3}}}\\
\qquad\qquad\quad - a^{\frac{2}{3}}b^{\frac{1}{3}} + b\\
\qquad\qquad\quad \underline{- a^{\frac{2}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}}}\\
\qquad\qquad\qquad\quad a^{\frac{1}{3}}b^{\frac{2}{3}} + b\\
\qquad\qquad\qquad\quad \underline{a^{\frac{1}{3}}b^{\frac{2}{3}} + b}
\end{array}$$

5. $x^{\frac{1}{3}}+y^{\frac{1}{3}})\,x+y\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}$, etc.

$$\begin{array}{l} x+x^{\frac{2}{3}}y^{\frac{1}{3}} \\ \hline -x^{\frac{2}{3}}y^{\frac{1}{3}}+y, \text{ etc.} \end{array}$$

6. $m^{\frac{1}{5}}-n^{\frac{1}{5}})\,m-n\,(m^{\frac{4}{5}}+m^{\frac{3}{5}}n^{\frac{1}{5}}$, etc.

$$\begin{array}{l} m-m^{\frac{4}{5}}n^{\frac{1}{5}} \\ \hline m^{\frac{4}{5}}n^{\frac{1}{5}}-n, \text{ etc.} \end{array}$$

7. $x^{\frac{1}{4}}-3y^{\frac{1}{4}})\,x-81y\,(x^{\frac{3}{4}}+3x^{\frac{1}{2}}y^{\frac{1}{4}}$, etc.

$$\begin{array}{l} x-3x^{\frac{3}{4}}y^{\frac{1}{4}} \\ \hline 3x^{\frac{3}{4}}y^{\frac{1}{4}}-81y, \text{ etc.} \end{array}$$

8. $3a^{\frac{1}{4}}-2b^{\frac{1}{4}})\,81a-16b\,(27a^{\frac{3}{4}}+18a^{\frac{1}{2}}b^{\frac{1}{4}}$, etc.

$$\begin{array}{l} 81a-54a^{\frac{3}{4}}b^{\frac{1}{4}} \\ \hline 54a^{\frac{3}{4}}b^{\frac{1}{4}}-16b, \text{ etc.} \end{array}$$

9. $a^{\frac{1}{2}}+x^{\frac{1}{2}})\,a-x\,(a^{\frac{1}{2}}-x^{\frac{1}{2}}$

$$\begin{array}{l} a+a^{\frac{1}{2}}x^{\frac{1}{2}} \\ \hline -a^{\frac{1}{2}}x^{\frac{1}{2}}-x \\ -a^{\frac{1}{2}}x^{\frac{1}{2}}-x \\ \hline \end{array}$$

10. $m^{\frac{1}{5}}-3)\,m-243\,(m^{\frac{4}{5}}+3m^{\frac{3}{5}}$, etc.

$$\begin{array}{l} m-3m^{\frac{4}{5}} \\ \hline 3m^{\frac{4}{5}}-243 \\ 3m^{\frac{4}{5}}-9m^{\frac{3}{5}}, \text{ etc.} \\ \hline \end{array}$$

11. $x^{\frac{1}{2}}+7)\,x+17x^{\frac{1}{2}}+70\,(x^{\frac{1}{2}}+10$

$$\begin{array}{l} x+7x^{\frac{1}{2}} \\ \hline 10x^{\frac{1}{2}}+70 \\ 10x^{\frac{1}{2}}+70 \\ \hline \end{array}$$

12. $x^{\frac{1}{3}}-3)\,x^{\frac{2}{3}}+x^{\frac{1}{3}}-12\,(x^{\frac{1}{3}}+4$

$$\begin{array}{l} x^{\frac{2}{3}}-3x^{\frac{1}{3}} \\ \hline 4x^{\frac{1}{3}}-12 \\ 4x^{\frac{1}{3}}-12 \\ \hline \end{array}$$

13. $b^{\frac{1}{3}}-1)\,-b^{\frac{4}{3}}+3b-3b^{\frac{2}{3}}+b^{\frac{1}{3}}\,(-b+2b^{\frac{2}{3}}-b^{\frac{1}{3}}$

$$\begin{array}{l} -b^{\frac{4}{3}}+b \\ \hline 2b-3b^{\frac{2}{3}} \\ 2b-2b^{\frac{2}{3}} \\ \hline -b^{\frac{2}{3}}+b^{\frac{1}{3}} \\ -b^{\frac{2}{3}}+b^{\frac{1}{3}} \\ \hline \end{array}$$

14. $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})\,x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}\,(x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}-x^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}+z^{\frac{2}{3}}$

$\underline{x+x^{\frac{2}{3}}y^{\frac{1}{3}}+x^{\frac{2}{3}}z^{\frac{1}{3}}}$

$-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{2}{3}}z^{\frac{1}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+y+z$

$\underline{-x^{\frac{2}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}}$

$x^{\frac{1}{3}}y^{\frac{2}{3}}+y-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}+z$

$\underline{x^{\frac{1}{3}}y^{\frac{2}{3}}+y+y^{\frac{2}{3}}z^{\frac{1}{3}}}$

$-x^{\frac{2}{3}}z^{\frac{1}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+z$

$\underline{-x^{\frac{2}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-x^{\frac{1}{3}}z^{\frac{2}{3}}}$

$-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}+x^{\frac{1}{3}}z^{\frac{2}{3}}+z$

$\underline{-x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{2}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{2}{3}}}$

$x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z$

$\underline{x^{\frac{1}{3}}z^{\frac{2}{3}}+y^{\frac{1}{3}}z^{\frac{2}{3}}+z}$

15. $x^{\frac{1}{3}}+4)\,x-5x^{\frac{2}{3}}-46x^{\frac{1}{3}}-40\,(x^{\frac{2}{3}}-9x^{\frac{1}{3}}-10.$

$\underline{x+4x^{\frac{2}{3}}}$

$-9x^{\frac{2}{3}}-46x^{\frac{1}{3}}$

$\underline{-9x^{\frac{2}{3}}-36x^{\frac{1}{3}}}$

$-10x^{\frac{1}{3}}-40$

$\underline{-10x^{\frac{1}{3}}-40}$

16. $m^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}})\,m+m^{\frac{1}{2}}n^{\frac{1}{2}}+n\,(m^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{1}{4}}+n^{\frac{1}{2}}$

$\underline{m-m^{\frac{3}{4}}n^{\frac{1}{4}}+m^{\frac{1}{2}}n^{\frac{1}{2}}}$

$m^{\frac{3}{4}}n^{\frac{1}{4}}+n$

$\underline{m^{\frac{3}{4}}n^{\frac{1}{4}}-m^{\frac{1}{2}}n^{\frac{1}{2}}+m^{\frac{1}{4}}n^{\frac{3}{4}}}$

$m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n$

$\underline{m^{\frac{1}{2}}n^{\frac{1}{2}}-m^{\frac{1}{4}}n^{\frac{3}{4}}+n}$

17. $p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1)\,p - 4p^{\frac{3}{4}} + 6p^{\frac{1}{2}} - 4p^{\frac{1}{4}} + 1\,(p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1$

$$\begin{array}{l}
\underline{p - 2p^{\frac{3}{4}} + p^{\frac{1}{2}}} \\
\quad - 2p^{\frac{3}{4}} + 5p^{\frac{1}{2}} - 4p^{\frac{1}{4}} \\
\quad \underline{- 2p^{\frac{3}{4}} + 4p^{\frac{1}{2}} - 2p^{\frac{1}{4}}} \\
\qquad\qquad p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1 \\
\qquad\qquad \underline{p^{\frac{1}{2}} - 2p^{\frac{1}{4}} + 1}
\end{array}$$

18. $2x^{\frac{1}{2}} + 3y^{\frac{1}{2}} + z^{\frac{1}{2}})\,2x + x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - x^{\frac{1}{2}}z^{\frac{1}{2}} - z\,(x^{\frac{1}{2}} - y^{\frac{1}{2}} - z^{\frac{1}{2}}$

$$\begin{array}{l}
\underline{2x + 3x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{2}}z^{\frac{1}{2}}} \\
\quad - 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - 4y^{\frac{1}{2}}z^{\frac{1}{2}} - 2x^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
\quad \underline{- 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3y - y^{\frac{1}{2}}z^{\frac{1}{2}}} \\
\qquad\qquad - 2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z \\
\qquad\qquad \underline{- 2x^{\frac{1}{2}}z^{\frac{1}{2}} - 3y^{\frac{1}{2}}z^{\frac{1}{2}} - z}
\end{array}$$

19. $x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}})\,x + y\,(x^{\frac{1}{5}} + y^{\frac{1}{5}}$

$$\begin{array}{l}
\underline{x - x^{\frac{4}{5}}y^{\frac{1}{5}} + x^{\frac{3}{5}}y^{\frac{2}{5}} - x^{\frac{2}{5}}y^{\frac{3}{5}} + x^{\frac{1}{5}}y^{\frac{4}{5}}} \\
\quad x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y \\
\quad \underline{x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y}
\end{array}$$

CV.

1.
$$\begin{array}{l}
a^{-1} + b^{-1} \\
\underline{a^{-1} - b^{-1}} \\
a^{-2} + a^{-1}b^{-1} \\
\underline{\qquad - a^{-1}b^{-1} - b^{-2}} \\
a^{-2} \qquad\qquad - b^{-2}
\end{array}$$

2.
$$\begin{array}{l}
x^{-3} + b^{-2} \\
\underline{x^{-3} - b^{-2}} \\
x^{-6} + x^{-3}b^{-2} \\
\underline{\qquad - x^{-3}b^{-2} - b^{-4}} \\
x^{-6} \qquad\qquad - b^{-4}
\end{array}$$

3.
$$\begin{array}{l}
x^{3}+x+x^{-1}+x^{-3} \\
x-x^{-1} \\
\hline
x^{4}+x^{2}+1+x^{-2} \\
\quad -x^{2}-1-x^{-2}-x^{-4} \\
\hline
x^{4} \qquad\qquad\qquad -x^{-4}
\end{array}$$

4.
$$\begin{array}{l}
x^{2}-1+x^{-2} \\
x^{2}+1+x^{-2} \\
\hline
x^{4}-x^{2}+1 \\
\quad +x^{2}-1+x^{-2} \\
\qquad\quad +1-x^{-2}+x^{-4} \\
\hline
x^{4} \quad\; +1+x^{-4}
\end{array}$$

5.
$$\begin{array}{l}
a^{-2}+b^{-2} \\
a^{-2}-b^{-2} \\
\hline
a^{-4}+a^{-2}b^{-2} \\
\qquad -a^{-2}b^{-1}-b^{-4} \\
\hline
a^{-4} \qquad\qquad -b^{-4}
\end{array}$$

6.
$$\begin{array}{l}
a^{-1}-b^{-1}+c^{-1} \\
a^{-1}+b^{-1}+c^{-1} \\
\hline
a^{-2}-a^{-1}b^{-1}+a^{-1}c^{-1} \\
\qquad +a^{-1}b^{-1}-b^{-2}+b^{-1}c^{-1} \\
\qquad\qquad\qquad +a^{-1}c^{-1}-b^{-1}c^{-1}+c^{-2} \\
\hline
a^{-2}+2a^{-1}c^{-1}-b^{-2}+c^{-2}
\end{array}$$

7.
$$\begin{array}{l}
1+ab^{-1}+a^{2}b^{-2} \\
1-ab^{-1}+a^{2}b^{-2} \\
\hline
1+ab^{-1}+a^{2}b^{-2} \\
\quad -ab^{-1}-a^{2}b^{-2}-a^{3}b^{-3} \\
\qquad\qquad +a^{2}b^{-2}+a^{3}b^{-3}+a^{4}b^{-4} \\
\hline
1 \qquad\qquad +a^{2}b^{-2}+a^{4}b^{-4}
\end{array}$$

8.
$$\begin{array}{l}
a^{2}b^{-2}+2+a^{-2}b^{2} \\
a^{2}b^{-2}-2-a^{-2}b^{2} \\
\hline
a^{4}b^{-4}+2a^{2}b^{-2}+1 \\
\qquad -2a^{2}b^{-2}-4-2a^{-2}b^{2} \\
\qquad\qquad\qquad -1-2a^{-2}b^{2}-a^{-4}b^{4} \\
\hline
a^{4}b^{-4} \quad 4 \quad -a^{-4}b^{4}-4a^{-2}b^{2}
\end{array}$$

9.
$$\begin{array}{l}
4x^{-3}+3x^{-2}+2x^{-1}+1 \\
x^{-2}-x^{-1}+1 \\
\hline
4x^{-5}+3x^{-4}+2x^{-3}+x^{-2} \\
\qquad -4x^{-4}-3x^{-3}-2x^{-2}-x^{-1} \\
\qquad\qquad +4x^{-3}+3x^{-2}+2x^{-1}+1 \\
\hline
4x^{-5}-x^{-4}+3x^{-3}+2x^{-2}+x^{-1}+1
\end{array}$$

10.
$$\begin{array}{l}
\frac{5}{2}x^{-2}+3x^{-1}-\frac{7}{3} \\
2x^{-2}-x^{-1}-\frac{1}{2} \\
\hline
5x^{-4}+6x^{-3}-\frac{14}{3}x^{-2} \\
\qquad -\frac{5}{2}x^{-3}-3x^{-2}+\frac{7}{3}x^{-1} \\
\qquad\qquad -\frac{5}{4}x^{-2}-\frac{3}{2}x^{-1}+\frac{7}{6} \\
\hline
5x^{-4}+\frac{7}{2}x^{-3}-\frac{107}{12}x^{-2}+\frac{5}{6}x^{-1}+\frac{7}{6}
\end{array}$$

CVI.

1.
$$\begin{array}{l}
x+x^{-1})\,x^2-x^{-2}\,(x-x^{-1} \\
\qquad\quad x^2+1 \\
\qquad\quad \overline{\quad -1-x^{-2}} \\
\qquad\qquad -1-x^{-2} \\
\qquad\qquad \overline{\qquad\qquad}
\end{array}$$

2.
$$\begin{array}{l}
a-b^{-1})\,a^2-\;b^{-2}\,(a+b^{-1} \\
\qquad\quad a^2-ab^{-1} \\
\qquad\quad \overline{\qquad\quad ab^{-1}-b^{-2}} \\
\qquad\qquad\qquad ab^{-1}-b^{-2} \\
\qquad\qquad\qquad \overline{\qquad\qquad}
\end{array}$$

3.
$$\begin{array}{l}
m+n^{-1})\,m^3+n^{-3}\,(m^2-mn^{-1}+n^{-2} \\
\qquad\quad m^3+m^2n^{-1} \\
\qquad\quad \overline{\quad -m^2n^{-1}+n^{-3}} \\
\qquad\qquad -m^2n^{-1}-mn^{-2} \\
\qquad\qquad \overline{\qquad\qquad mn^{-2}+n^{-3}} \\
\qquad\qquad\qquad\quad mn^{-2}+n^{-3} \\
\qquad\qquad\qquad\quad \overline{\qquad\qquad}
\end{array}$$

4. $c-d^{-1})\,c^5-d^{-5}\,(c^4+c^3d^{-1}+c^2d^{-2}$, etc.

$$\begin{array}{l}
\underline{c^5-c^4d^{-1}} \\
\quad c^4d^{-1}-d^{-5} \\
\quad \underline{c^4d^{-1}-c^3d^{-2}} \\
\qquad c^3d^{-2}-d^{-5} \\
\qquad \underline{c^3d^{-2}-c^2d^{-3}}\text{, etc.}
\end{array}$$

5. $xy^{-1}+x^{-1}y)\,x^2y^{-2}+2+x^{-2}y^2\,(xy^{-1}+x^{-1}y$

$$\begin{array}{l}
\underline{x^2y^{-2}+1} \\
\qquad 1+x^{-2}y^2 \\
\qquad \underline{1+x^{-2}y^2}
\end{array}$$

6. $a^{-2}-a^{-1}b^{-1}+b^{-2})\,a^{-4}+a^{-2}b^{-2}+b^{-4}\,(a^{-2}+a^{-1}b^{-1}+b^{-2}$

$$\begin{array}{l}
\underline{a^{-4}-a^{-3}b^{-1}+a^{-2}b^{-2}} \\
\quad a^{-3}b^{-1}+b^{-4} \\
\quad \underline{a^{-3}b^{-1}-a^{-2}b^{-2}+a^{-1}b^{-3}} \\
\qquad a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4} \\
\qquad \underline{a^{-2}b^{-2}-a^{-1}b^{-3}+b^{-4}}
\end{array}$$

7. $xy^{-1}-x^{-1}y)\,x^3y^{-3}-3xy^{-1}+3x^{-1}y-x^{-3}y^3\,(x^2y^{-2}-2+x^{-2}y^2$

$$\begin{array}{l}
\underline{x^3y^{-3}-\ xy^{-1}} \\
\quad -2xy^{-1}+3x^{-1}y \\
\quad \underline{-2xy^{-1}+2x^{-1}y} \\
\qquad x^{-1}y-x^{-3}y^3 \\
\qquad \underline{x^{-1}y-x^{-3}y^3}
\end{array}$$

8.
$$\begin{array}{l}
\frac{x^{-2}}{2}-x^{-1}+3\Big)\frac{3x^{-5}}{4}-4x^{-4}+\frac{77x^{-3}}{8}-\frac{43x^{-2}}{4}-\frac{33x^{-1}}{4}+27\Big(\frac{3x^{-3}}{2}-5x^{-2}+\frac{x^{-1}}{4}+9 \\
\qquad\qquad\qquad \underline{\frac{3x^{-5}}{4}-\frac{3x^{-4}}{2}+\frac{9x^{-3}}{2}} \\
\qquad\qquad\qquad\qquad -\frac{5x^{-4}}{2}+\frac{41x^{-3}}{8}-\frac{43x^{-2}}{4} \\
\qquad\qquad\qquad\qquad \underline{-\frac{5x^{-4}}{2}+5x^{-3}-15x^{-2}} \\
\qquad\qquad\qquad\qquad\qquad\qquad \frac{x^{-3}}{8}+\frac{17x^{-2}}{4}-\frac{33x^{-1}}{4} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{x^{-3}}{8}-\frac{x^{-2}}{4}+\frac{3x^{-1}}{4}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{9x^{-2}}{2}-9x^{-1}+27 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\frac{9x^{-2}}{2}-9x^{-1}+27}
\end{array}$$

9.
$$\begin{array}{l}
ab^{-1}+a^{-1}b)a^3b^{-3}+a^{-3}b^3(a^2b^{-2}-1+a^{-2}b^2 \\
\qquad\qquad\quad \underline{a^3b^{-3}+ab^{-1}} \\
\qquad\qquad\qquad -ab^{-1}+a^{-3}b^3 \\
\qquad\qquad\qquad \underline{-ab^{-1}-a^{-1}b} \\
\qquad\qquad\qquad\qquad a^{-1}b+a^{-3}b^3 \\
\qquad\qquad\qquad\qquad \underline{a^{-1}b+a^{-3}b^3}
\end{array}$$

0. $a^{-1}+b^{-1}+c^{-1})a^{-3}+b^{-3}+c^{-3}-3a^{-1}b^{-1}c^{-1}(a^{-2}-a^{-1}b^{-1}-a^{-1}c^{-1}$

$$\begin{array}{l}
\underline{a^{-3}+a^{-2}b^{-1}+a^{-2}c^{-1}} \qquad\qquad +b^{-2}-b^{-1}c^{-1}+c^{-2} \\
-a^{-2}b^{-1}-a^{-2}c^{-1}-3a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3} \\
\underline{-a^{-2}b^{-1}-a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}} \\
-a^{-2}c^{-1}+a^{-1}b^{-2}-2a^{-1}b^{-1}c^{-1}+b^{-3}+c^{-3} \\
\underline{-a^{-2}c^{-1}-a^{-1}b^{-1}c^{-1}-a^{-1}c^{-2}} \\
a^{-1}b^{-2}-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}+b^{-3}+c^{-3} \\
\underline{a^{-1}b^{-2}+b^{-3}+b^{-2}c^{-1}} \\
-a^{-1}b^{-1}c^{-1}+a^{-1}c^{-2}-b^{-2}c^{-1}+c^{-3} \\
\underline{-a^{-1}b^{-1}c^{-1}-b^{-2}c^{-1}-b^{-1}c^{-2}} \\
a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3} \\
\underline{a^{-1}c^{-2}+b^{-1}c^{-2}+c^{-3}}
\end{array}$$

CVII.

1. $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y)x^{\frac{4}{3}}-4xy+4x^{\frac{2}{3}}y+4y^2(x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y$

$$\begin{array}{l}
\underline{x^{\frac{4}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2x^{\frac{2}{3}}y} \\
-2x^{\frac{1}{3}}y^{\frac{1}{2}}-4xy+2x^{\frac{2}{3}}y \\
\underline{-2x^{\frac{1}{3}}y^{\frac{1}{2}}-4xy-4x^{\frac{1}{3}}y^{\frac{3}{2}}} \\
2x^{\frac{2}{3}}y+4x^{\frac{1}{3}}y^{\frac{3}{2}}+4y^2 \\
\underline{2x^{\frac{2}{3}}y+4x^{\frac{1}{3}}y^{\frac{3}{2}}+4y^2}
\end{array}$$

2. $\{x^{15ab}\times x^{12}\}^{\frac{1}{3a-2}}=x^{\frac{15ab+12}{3a-2}}$.

3. $(x^{10b+18a})^{\frac{1}{3a-2}}=x^{\frac{10b+18a}{3a-2}}$.

4. $\left\{\frac{x^2+a^2-x^2+a^2}{x^4-a^4}-\frac{a(x-a-x-a)}{x^4-a^4}\right\}=\left(\frac{4a^2}{x^4-a^4}\right)^{\frac{1}{2}}$, etc.

5. $$\begin{array}{l}
\frac{7}{3}x^{-2}+4x^{-1}-\frac{2}{7} \\
3x^{-2}-2x^{-1}-\frac{1}{2} \\
\hline
7x^{-4}+12x^{-3}-\frac{6}{7}x^{-2} \\
\qquad -\frac{14}{3}x^{-3}-8x^{-2}+\frac{4}{7}x^{-1} \\
\qquad\qquad -\frac{7}{6}x^{-2}-2x^{-1}+\frac{1}{7} \\
\hline
7x^{-4}+\frac{22}{3}x^{-3}-\frac{421}{42}x^{-2}-\frac{10}{7}x^{-1}+\frac{1}{7}
\end{array}$$

6. $$\frac{x^{a+b+a-b+c-2a}}{x^{c-a}}=x^{c-c+a}=x^{a}.$$

7. $$\begin{array}{l}
x^n+y^n)\,x^{2n}-y^{2n}\,(x^n-y^n \\
\qquad\quad x^{2n}+x^ny^n \\
\hline
\qquad\qquad -x^ny^n-y^{2n} \\
\qquad\qquad -x^ny^n-y^{2n} \\
\hline
\end{array}$$

8. $$\begin{array}{l}
a^{\frac{3}{2}}+3ab^{\frac{2}{3}}+3a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{6}{3}} \\
a^{\frac{1}{2}}-b^{\frac{2}{3}} \\
\hline
a^2+3a^{\frac{3}{2}}b^{\frac{2}{3}}+3ab^{\frac{4}{3}}+a^{\frac{1}{2}}b^{\frac{6}{3}} \\
\quad -\ a^{\frac{3}{2}}b^{\frac{2}{3}}-3ab^{\frac{4}{3}}-3a^{\frac{1}{2}}b^{\frac{6}{3}}-b^{\frac{8}{3}} \\
\hline
a^2+2a^{\frac{3}{2}}b^{\frac{2}{3}}\qquad\qquad -2a^{\frac{1}{2}}b^{\frac{6}{3}}-b^{\frac{8}{3}}
\end{array}$$

9. $$\begin{array}{l}
a^{\frac{1}{3}}-b^{\frac{1}{3}})\,a-b\,(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}},\ \text{etc.} \\
\qquad\quad a-a^{\frac{2}{3}}b^{\frac{1}{3}} \\
\hline
\qquad\qquad a^{\frac{2}{3}}b^{\frac{1}{3}}-b,\ \text{etc.}
\end{array}$$

10. $(a^2)^m = a^{2+2+\cdots \quad m\text{ terms}} = a^{2m} = a^m \times a^m = (a^m)^2.$

11. $a^{m^n} = a^{mn}$; $\therefore m^n = mn$; $m^{n-1} = n$; $m = n^{\frac{1}{n-1}}.$

12. $x^{a+b+c+a+b-c+a-b+c+b+c-a} = x^{2a+2b+2c}.$

13. $(x^p)^p \div (x^p)^{p-q} = x^{p^2} \div x^{p^2-pq} = x^{pq}.$

14. $4a^x \div \dfrac{1}{4a^x} = 4a^x \times 4a^x = 16a^{2x}.$

15. $[\{a^{mn}\}^p] \div [\{a^{mn}\}^{-p}] = a^{mnp} \div a^{-mnp} = a^{2mnp}.$

16. $2a^{2m} + 2a^m b^p - 4a^m c^n - 3a^m b - 3b^{p+1} + 6bc^n.$

17. $a^{m-n+n-m}.\quad b^{n-p+p-n}.\quad c = 1 \times 1 \times c = c.$

18. $$\frac{a^{\frac{1}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}})}{(a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})} = \frac{a^{\frac{1}{3}}}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}.$$

19. $(x^{\frac{1}{3}} + x^{\frac{1}{6}} + 1)(x^{\frac{1}{3}} - x^{\frac{1}{6}} + 1) = x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$
$(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1) = x^{\frac{4}{3}} + x^{\frac{2}{3}} + 1$

20.
$$\begin{array}{l}
a^m - ba^{m-1}x + ca^{m-2}x^2 \\
\underline{a^n + ba^{n-1}x - ca^{n-2}x^2} \\
a^{m+n} - ba^{m+n-1}x + ca^{m+n-2}x^2 \\
\qquad\quad + ba^{m+n-1}x - b^2a^{m+n-2}x^2 + bca^{m+n-3}x^3 \\
\qquad\qquad\qquad\quad \underline{- ca^{m+n-2}x^2 + bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4} \\
a^{m+n} - b^2a^{m+n-2}x^2 + 2bca^{m+n-3}x^3 - c^2a^{m+n-4}x^4
\end{array}$$

21.
$$\begin{array}{l}
x^{pq-p} + y^{pq-q})\, x^{2pq-2p} - y^{2pq-2q}\, (x^{pq-p} - y^{pq-q} \\
\qquad\qquad\qquad\quad \underline{x^{2pq-2p} + x^{pq-p}y^{pq-q}} \\
\qquad\qquad\qquad\qquad\quad - x^{pq-p}y^{pq-q} - y^{2pq-2q} \\
\qquad\qquad\qquad\qquad\quad \underline{- x^{pq-p}y^{pq-q} - y^{2pq-2q}}
\end{array}$$

22. $\left\{(a^m)^{\frac{m^2-1}{m}}\right\}^{\frac{1}{m+1}} = a^{\frac{m^2-1}{m+1}} = a^{m-1}.$

23. $$\begin{array}{l} x^{3r}+x^{2r}y^p+x^ry^{2p}+y^{3p} \\ x^r-y^p \\ \hline x^{4r}+x^{3r}y^p+x^{2r}y^{2p}+x^ry^{3p} \\ \quad -x^{3r}y^p-x^{2r}y^{2p}-x^ry^{3p}-y^{4p} \\ \hline x^{4r} \qquad\qquad\qquad\qquad\qquad -y^{4p} \end{array}$$

24. $\sqrt[4]{625}=\sqrt{25}=5$; and $\frac{1}{12^2}=\frac{1}{144}$.

25. $$\begin{array}{l} x^{mn-n}-y^{mn-m} \\ x^n-y^m \\ \hline x^{mn}-x^ny^{mn-m}-x^{mn-n}y^m+y^{mn} \end{array}$$

26. $$\begin{array}{l} x^{\frac{3}{4}}+3x^{\frac{1}{2}}-1 \\ x^{\frac{1}{4}}-2x^{-\frac{1}{4}} \\ \hline x+3x^{\frac{3}{4}}-x^{\frac{1}{4}} \\ \quad -2x^{\frac{1}{2}}-6x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \\ \hline x+3x^{\frac{3}{4}}-2x^{\frac{1}{2}}-7x^{\frac{1}{4}}+2x^{-\frac{1}{4}} \end{array}$$

CVIII.

1. $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$; $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; $\sqrt[6]{x^3}$ and $\sqrt[6]{y^2}$.

2. $4^{\frac{1}{3}}$ and $2^{\frac{1}{5}}$; $4^{\frac{5}{15}}$ and $2^{\frac{3}{15}}$; $\sqrt[15]{4^5}$ and $\sqrt[15]{2^3}$, etc.

3. $18^{\frac{1}{2}}$ and $50^{\frac{1}{3}}$; $18^{\frac{3}{6}}$ and $50^{\frac{2}{6}}$; $\sqrt[6]{18^3}$ and $\sqrt[6]{50^2}$, etc.

4. $2^{\frac{1}{m}}$ and $2^{\frac{1}{n}}$; $2^{\frac{n}{mn}}$ and $2^{\frac{m}{mn}}$; $\sqrt[mn]{2^n}$ and $\sqrt[mn]{2^m}$.

5. $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$; $a^{\frac{n}{mn}}$ and $b^{\frac{m}{mn}}$; $\sqrt[mn]{a^n}$ and $\sqrt[mn]{b^m}$.

6. $(a+b)^{\frac{1}{2}}$ and $(a-b)^{\frac{1}{3}}$; $(a+b)^{\frac{3}{6}}$ and $(a-b)^{\frac{2}{6}}$, etc.

CIX.

1. $\sqrt{24} = \sqrt{(4 \times 6)} = 2\sqrt{6}$. 2. $\sqrt{50} = \sqrt{(25 \times 2)} = 5\sqrt{2}$.

3. $\sqrt{(4a^3)} = 2a\sqrt{a}$. 4. $\sqrt{(125a^4d^3)} = 5a^2d\sqrt{(5d)}$.

5. $\sqrt{(16 \times 2yz^3)} = 4z\sqrt{(2yz)}$. 6. $\sqrt{(100 \times 10a)} = 10\sqrt{(10a)}$.

7. $\sqrt{(144 \times 5c^2)} = 12c\sqrt{5}$. 8. $7.\sqrt{(36 \times 11x)} = 42\sqrt{(11x)}$.

9. $18.\sqrt{\left(\frac{1}{9} \times \frac{5}{3}x^3\right)} = 6x\sqrt{\frac{5x}{3}}$. 10. $a\sqrt{\left(a^2 \times \frac{a}{b}\right)} = a^2\sqrt{\frac{a}{b}}$.

11. $\sqrt{\{a(a^2 + 2ax + x^2)\}} = (a + x)\sqrt{a}$.

12. $\sqrt{\{x(x^2 - 2xy + y^2)\}} = (x - y)\sqrt{x}$.

13. $\sqrt{\{2(25a^2 - 50ab + 25b^2)\}} = 5(a - b)\sqrt{2}$.

14. $\sqrt{\{7y(9c^4 - 6c^2y + y^2)\}} = (3c^2 - y)\sqrt{(7y)}$.

15. $\sqrt[3]{(27a^6 \times 2b^2)} = 3a^2\sqrt[3]{(2b^2)}$.

16. $\sqrt[3]{(20xy \times 8x^3y^6)} = 2xy^2\sqrt[3]{(20xy)}$.

17. $\sqrt[3]{(27m^9n^9 \times 4n)} = 3m^3n^3\sqrt[3]{(4n)}$.

18. $\sqrt[3]{(343a^{15}b^{15} \times 4b)} = 7a^5b^5\sqrt[3]{(4b)}$.

19. $\sqrt[3]{\{x(x^3 + 3x^2y + 3xy^2 + y^3)\}} = (x + y)\sqrt[3]{x}$.

20. $\sqrt[3]{\{a(a^3 - 3a^2b + 3ab^2 - b^3)\}} = (a - b)\sqrt[3]{a}$.

CX.

1. $4\sqrt{3} = \sqrt{16}\,.\sqrt{3} = \sqrt{48}$. 2. $3\sqrt{7} = \sqrt{9}\,.\sqrt{7} = \sqrt{63}$.

3. $5\sqrt[3]{9} = \sqrt[3]{125}\,.\sqrt[3]{9} = \sqrt[3]{1125}$. 4. $2\sqrt[4]{6} = \sqrt[4]{16}\,.\sqrt[4]{6} = \sqrt[4]{96}$.

5. $3\sqrt[3]{\frac{3}{7}} = \sqrt[3]{27}\,.\sqrt[3]{\frac{3}{7}} = \sqrt[3]{\frac{81}{7}}$.

6. $3\sqrt{a} = \sqrt{9}\,.\sqrt{a} = \sqrt{(9a)}$.

7. $4a\sqrt{(3x)} = \sqrt{(16a^2)}\sqrt{(3x)} = \sqrt{(48a^2x)}$.

8. $2ax\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(4a^2x^2)}.\sqrt{\left(\frac{3a}{4x}\right)} = \sqrt{(3a^3x)}$.

9. $\sqrt{(m+n)^2}.\sqrt{\left(\frac{m-n}{m+n}\right)} = \sqrt{(m^2-n^2)}$.

10. $\sqrt{(a+b)^2}.\sqrt{\left(\frac{1}{a^2-b^2}\right)} = \sqrt{\frac{(a+b)^2}{a^2-b^2}} = \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}$.

11. $\sqrt{}.\left(\frac{x-y}{x+y}\right)^2.\sqrt{\left(\frac{x^2+xy}{x^2-2xy+y^2}\right)} = \sqrt{}.\frac{x^2+xy}{\underline{(x+y)^2}} = \left(\frac{x}{x+y}\right)^{\frac{1}{2}}$. ?

CXI.

1. $\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$; $\sqrt[3]{4} = \sqrt[6]{4^2} = \sqrt[6]{16}$, etc.

2. $\sqrt{10} = \sqrt[6]{10^3} = \sqrt[6]{1000}$; $\sqrt[3]{15} = \sqrt[6]{15^2} = \sqrt[6]{225}$, etc.

3. $2\sqrt{3} = \sqrt{(4\times 3)} = \sqrt{12}$; $3\sqrt{2} = \sqrt{(9\times 2)} = \sqrt{18}$, etc.

4. $\sqrt{\frac{3}{5}} = \sqrt[6]{\frac{27}{125}}$; . $\sqrt[3]{\frac{14}{15}} = \sqrt[6]{\frac{196}{225}}$, etc.

5. $3\sqrt{7} = \sqrt{(9\times 7)} = \sqrt{63}$; $4\sqrt{3} = \sqrt{(16\times 3)} = \sqrt{48}$, etc.

6. $2\sqrt{87} = \sqrt{(4\times 87)} = \sqrt{348}$; $3\sqrt{33} = \sqrt{(9\times 33)} = \sqrt{297}$, etc.

7. $2\sqrt[3]{22} = \sqrt[3]{176} = \sqrt[6]{30976}$; $3\sqrt[3]{7} = \sqrt[3]{189} = \sqrt[6]{35721}$;
$4\sqrt{2} = \sqrt{32} = \sqrt[6]{32768}$, etc.

8. $3\sqrt{19} = \sqrt{171} = \sqrt[6]{5000211}$; $5\sqrt[3]{18} = \sqrt[3]{2250} = \sqrt[6]{5062500}$;
$3\sqrt[3]{82} = \sqrt[3]{2214} = \sqrt[6]{4901796}$, etc.

9. $2\sqrt[3]{14} = \sqrt[3]{112}$; $5\sqrt[3]{2} = \sqrt[3]{250}$; $3\sqrt[3]{3} = \sqrt[3]{81}$, etc.

10. $\frac{1}{2}\sqrt{2} = \sqrt{\frac{1}{4}}$; $\frac{1}{3}\sqrt{3} = \sqrt{\frac{1}{9}}$; $\frac{1}{4}\sqrt{4} = \sqrt{\frac{1}{16}}$, etc.

CXII.

1. $3\sqrt{3}+8\sqrt{3}+18\sqrt{3}=29\sqrt{3}$.

2. $30\sqrt{10}+20\sqrt{2}+144\sqrt{2}=30\sqrt{10}+164\sqrt{2}$.

3. $a^2\sqrt{x}+b^2\sqrt{x}+c^2\sqrt{x}=(a^2+b^2+c^2)\sqrt{x}$.

4. $4\sqrt[3]{2}+7\sqrt[3]{2}+2\sqrt[3]{2}=13\sqrt[3]{2}$.

5. $21\sqrt[3]{2}+6\sqrt[3]{2}+6\sqrt[3]{2}=33\sqrt[3]{2}$.

6. $4\sqrt{6}-3\sqrt{6}=\sqrt{6}$.

7. $9\sqrt{3}-4\sqrt{3}=5\sqrt{3}$.

8. $72\sqrt{2}-24\sqrt{2}=48\sqrt{2}$.

9. $10\sqrt[3]{2}-6\sqrt[3]{2}=4\sqrt[3]{2}$.

10. $21\sqrt[3]{3}-21\sqrt[3]{3}=0$.

11. $\sqrt{6}\times\sqrt{8}=\sqrt{48}=4\sqrt{3}$.

12. $\sqrt{14}\times\sqrt{20}=\sqrt{280}=2\sqrt{70}$.

13. $\sqrt{50}\times\sqrt{200}=\sqrt{10000}=100$.

14. $\sqrt[3]{(3a^2b)}\times\sqrt[3]{(9ab^2)}=\sqrt[3]{(27a^3b^3)}=3ab$.

15. $\sqrt[3]{(12ab)}\times\sqrt[3]{(8a^2b^3)}=\sqrt[3]{(96a^3b^4)}=2ab\sqrt[3]{(12b)}$.

16. $\sqrt{12}\div\sqrt{3}=\sqrt{4}=2$.

17. $\sqrt{18}\div\sqrt{50}=\sqrt{\frac{9}{25}}=\frac{3}{5}$.

18. $\sqrt[3]{(a^2b)}\div\sqrt[3]{(ab^2)}=\sqrt[3]{\left(\frac{a}{b}\right)}$.

19. $\sqrt[4]{(a^3b)}\div\sqrt[4]{(ab^3)}=\sqrt[4]{\left(\frac{a^2}{b^2}\right)}=\sqrt{\left(\frac{a}{b}\right)}$.

20. $\sqrt{(x^2+x^3y)}\div\sqrt{(x+2x^2y+x^3y^2)}=\sqrt{\left(\frac{x+x^2y}{1+2xy+x^2y^2}\right)}=\sqrt{\frac{x}{1+xy}}$.

CXIII.

1. $\sqrt{x}\times\sqrt{y}=\sqrt{(xy)}$.

2. $\sqrt{(x-y)}\times\sqrt{y}=\sqrt{(xy-y^2)}$.

3. $\sqrt{(x+y)}\times\sqrt{(x+y)}=x+y$.

4. $\sqrt{(x-y)}\times\sqrt{(x+y)}=\sqrt{(x^2-y^2)}$.

11.
$$\begin{array}{l}\sqrt{x}+\sqrt{y}+\sqrt{z}\\ \sqrt{x}-\sqrt{y}+\sqrt{z}\\ \hline x+\sqrt{(xy)}+\sqrt{(xz)}\\ \quad -\sqrt{(xy)}-y-\sqrt{(yz)}\\ \qquad\qquad +\sqrt{(xz)}+\sqrt{(yz)}+z\\ \hline x-y+2\sqrt{(xz)}+z\end{array}$$

12.
$$\begin{array}{l}\sqrt{a}+\sqrt{(a-x)}+\sqrt{x}\\ \sqrt{a}-\sqrt{(a-x)}+\sqrt{x}\\ \hline a+\sqrt{(a^2-ax)}+\sqrt{(ax)}\\ \quad -\sqrt{(a^2-ax)}-(a-x)-\sqrt{(ax-x^2)}\\ \qquad\qquad\qquad +\sqrt{(ax)}+\sqrt{(ax-x^2)}+x\\ \hline 2x+2\sqrt{(ax)}\end{array}$$

13.
$$\begin{array}{l}21+\sqrt{(x^2-9)}\\ 21+\sqrt{(x^2-9)}\\ \hline 441+21\sqrt{(x^2-9)}\\ \qquad +21\sqrt{(x^2-9)}+x^2-9\\ \hline 432+42\sqrt{(x^2-9)}+x^2\end{array}$$

14.
$$\begin{array}{l}\sqrt{(x+3)}+\sqrt{(x+8)}\\ \sqrt{(x+3)}+\sqrt{(x+8)}\\ \hline x+3+\sqrt{(x^2+11x+24)}\\ \qquad +\sqrt{(x^2+11x+24)}+x+8\\ \hline 2x+11+2\sqrt{(x^2+11x+24)}\end{array}$$

15.
$$\begin{array}{l}\sqrt{x}+\sqrt{(x-4)}\\ \sqrt{x}+\sqrt{(x-4)}\\ \hline x+\sqrt{(x^2-4x)}\\ \quad +\sqrt{(x^2-4x)}+x-4\\ \hline 2x-4+2\sqrt{(x^2-4x)}\end{array}$$

16.
$$\begin{array}{l}\sqrt{(x-6)}+\sqrt{x}\\ \sqrt{(x-6)}+\sqrt{x}\\ \hline x-6+\sqrt{(x^2-6x)}\\ \qquad +\sqrt{(x^2-6x)}+x\\ \hline 2x-6+2\sqrt{(x^2-6x)}\end{array}$$

17.
$$\begin{array}{l} 2\sqrt{x}-3 \\ 2\sqrt{x}-3 \\ \hline 4x-6\sqrt{x} \\ \quad -6\sqrt{x}+9 \\ \hline 4x-12\sqrt{x}+9 \end{array}$$

18.
$$\begin{array}{l} \sqrt{(x+y)}-\sqrt{(x-y)} \\ \sqrt{(x+y)}-\sqrt{(x-y)} \\ \hline x+y-\sqrt{(x^2-y^2)} \\ \qquad -\sqrt{(x^2-y^2)}+x-y \\ \hline 2x-2\sqrt{(x^2-y^2)} \end{array}$$

19.
$$\begin{array}{l} \sqrt{x}\,.\sqrt{(x+1)}-\sqrt{(x-1)} \\ \sqrt{x}\,.\sqrt{(x+1)}-.\sqrt{(x-1)} \\ \hline x(x+1)-\sqrt{x}\,.\sqrt{(x^2-1)} \\ \qquad -\sqrt{x}\,.\sqrt{(x^2-1)}+(x-1) \\ \hline x^2+2x-1-2\sqrt{(x^3-x)} \end{array}$$

20.
$$\begin{array}{l} \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \sqrt{(x+1)}+\sqrt{x}.\sqrt{(x-1)} \\ \hline x+1+\sqrt{x}.\sqrt{(x^2-1)} \\ \qquad +\sqrt{x}.\sqrt{(x^2-1)}+x(x-1) \\ \hline x^2+1+2\sqrt{(x^3-x)} \end{array}$$

CXV.

1. $c-d=(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$.
2. $c^2-d=(c+\sqrt{d})(c-\sqrt{d})$.
3. $c-d^2=(\sqrt{c}+d)(\sqrt{c}-d)$.
4. $1-y=(1+\sqrt{y})(1-\sqrt{y})$.
5. $1-3x^2=(1+\sqrt{3}.x)(1-\sqrt{3}.x)$.
6. $5m^2-1=(\sqrt{5}.m+1)(\sqrt{5}.m-1)$.
7. $4a^2-3x=\{2a+\sqrt{(3x)}\}\{2a-\sqrt{(3x)}\}$.
8. $9-8n=\{3+2\sqrt{(2n)}\}\{3-2\sqrt{(2n)}\}$.
9. $11n^2-16=(\sqrt{11}.n+4)(\sqrt{11}.n-4)$.
10. $p^2-4r=(p+2\sqrt{r})(p-2\sqrt{r})$.
11. $p-3q^2=(\sqrt{p}+\sqrt{3}.q)(\sqrt{p}-\sqrt{3}.q)$.

12. $a^{2m}-b^{n}=\left(a^{m}+b^{\frac{n}{2}}\right)\left(a^{m}-b^{\frac{n}{2}}\right).$

13. $$\frac{1}{a-\sqrt{b}}\times\frac{a+\sqrt{b}}{a+\sqrt{b}}=\frac{a+\sqrt{b}}{a^2-b}.$$

14. $$\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{a+\sqrt{(ab)}}{a-b}.$$

15. $$\frac{4+3\sqrt{2}}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{12+17\sqrt{2}+12}{9-8}=24+17\sqrt{2}.$$

16. $$\frac{2}{2-\sqrt{2}}\times\frac{2+\sqrt{2}}{2+\sqrt{2}}=\frac{4+2\sqrt{2}}{4-2}=\frac{4+2\sqrt{2}}{2}=2+\sqrt{2}.$$

17. $$\frac{\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\frac{2\sqrt{3}+3}{4-3}=3+2\sqrt{3}.$$

18. $$\frac{2-\sqrt{2}}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}=\frac{4-4\sqrt{2}+2}{4-2}=\frac{6-4\sqrt{2}}{2}=3-2\sqrt{2}.$$

19. $$\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\times\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}}=\frac{a+2\sqrt{(ax)}+x}{a-x}.$$

20. $$\frac{1+\sqrt{x}}{1-\sqrt{x}}\times\frac{1+\sqrt{x}}{1+\sqrt{x}}=\frac{1+2\sqrt{x}+x}{1-x}.$$

21. $$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}\times\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}}$$
$$=\frac{a+x+2\sqrt{(a^2-x^2)}+a-x}{a+x-(a-x)}=\frac{2a+2\sqrt{(a^2-x^2)}}{2x}=\frac{a+\sqrt{(a^2-x^2)}}{x}.$$

22. $$\frac{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}{\sqrt{(m^2+1)}+\sqrt{(m^2-1)}}\times\frac{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}{\sqrt{(m^2+1)}-\sqrt{(m^2-1)}}$$
$$=\frac{m^2+1-2\sqrt{(m^4-1)}+m^2-1}{m^2+1-(m^2-1)}=\frac{2m^2-2\sqrt{(m^4-1)}}{2}$$
$$=m^2-\sqrt{(m^4-1)}.$$

23. $$\frac{a+\sqrt{(a^2-1)}}{a-\sqrt{(a^2-1)}}\times\frac{a+\sqrt{(a^2-1)}}{a+\sqrt{(a^2-1)}}=\frac{a^2+2a\sqrt{(a^2-1)}+a^2-1}{a^2-(a^2-1)}$$
$$=2a^2-1+2a\sqrt{(a^2-1)}.$$

24. $$\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}\times\frac{a+\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}=\frac{a^2+2a\sqrt{(a^2-x^2)}+a^2-x^2}{a^2-(a^2-x^2)}$$
$$=\frac{2a^2-x^2+2a\sqrt{(a^2-x^2)}}{x^2}.$$

CXVI.

1. $$\begin{array}{l} 4+\sqrt{(-3)}\\ 4-\sqrt{(-3)}\\ \hline 16+4\sqrt{(-3)}\\ \qquad -4\sqrt{(-3)}-(-3)\\ \hline 19 \end{array}$$

2. $$\begin{array}{l} \sqrt{3}-2\sqrt{(-2)}\\ \sqrt{3}+2\sqrt{(-2)}\\ \hline 3-2\sqrt{(-6)}\\ \qquad +2\sqrt{(-6)}-4\times(-2)\\ \hline 11 \end{array}$$

3. $$\begin{array}{l} 4\sqrt{(-2)}-2\sqrt{2}\\ \frac{1}{2}\sqrt{(-2)}-3\sqrt{2}\\ \hline 2\times(-2)-\sqrt{(-4)}\\ \qquad\qquad -12\sqrt{(-4)}+6\times 2\\ \hline \qquad 8-13\sqrt{(-4)}=8-26\sqrt{(-1)} \end{array}$$

4. $$\begin{array}{l} \sqrt{(-2)}+\sqrt{(-3)}+\sqrt{(-4)}\\ \sqrt{(-2)}-\sqrt{(-3)}-\sqrt{(-4)}\\ \hline -2+\sqrt{6}+2\sqrt{2}\\ \qquad -\sqrt{6}-(-3)-2\sqrt{3}\\ \qquad\qquad -2\sqrt{2}-2\sqrt{3}-(-4)\\ \hline -2+3-4\sqrt{3}+4=5-4\sqrt{3} \end{array}$$

5. $$\begin{array}{l} 3\sqrt{(-a)}+\sqrt{(-b)}\\ 4\sqrt{(-a)}-2\sqrt{(-b)}\\ \hline -12a+4\sqrt{(ab)}\\ \qquad -6\sqrt{(ab)}+2b\\ \hline -12a-2\sqrt{(ab)}+2b \end{array}$$

6. $$\begin{array}{l} a+\sqrt{(-a)}\\ a-\sqrt{(-a)}\\ \hline a^2+a\sqrt{(-a)}\\ \qquad -a\sqrt{(-a)}+a\\ \hline a^2+a \end{array}$$

7.
$$\begin{array}{lr} a\sqrt{(-a)}+b\sqrt{(-b)} & \\ a\sqrt{(-a)}-b\sqrt{(-b)} & \\ \hline -a^3+ab\sqrt{(ab)} & \\ \quad -ab\sqrt{(ab)}+b^3 & \\ \hline -a^3 \qquad\qquad +b^3 & \end{array}$$

8.
$$\begin{array}{l} a+\beta\sqrt{(-1)} \\ a-\beta\sqrt{(-1)} \\ \hline a^2+a\beta\sqrt{(-1)} \\ \quad -a\beta\sqrt{(-1)}+\beta^2 \\ \hline a^2 \qquad\qquad +\beta^2 \end{array}$$

9.
$$\begin{array}{l} 1-\sqrt{(1-e^2)} \\ 1+\sqrt{(1-e^2)} \\ \hline 1-\sqrt{(1-e^2)} \\ \quad +\sqrt{(1-e^2)}-(1-e^2) \\ \hline \qquad\qquad e^2 \end{array}$$

10.
$$\begin{array}{l} e^{p\sqrt{(-1)}}+e^{-p\sqrt{(-1)}} \\ e^{p\sqrt{(-1)}}-e^{-p\sqrt{(-1)}} \\ \hline e^{2p\sqrt{(-1)}}+1 \\ \qquad\qquad -1-e^{-2p\sqrt{(-1)}} \\ \hline e^{2p\sqrt{(-1)}} \qquad -e^{-2p\sqrt{(-1)}} \end{array}$$

CXVII.

1. $\dfrac{3x+3\sqrt{(xy)}-3\sqrt{(xy)}+3y}{9\sqrt{(xy)}}=\dfrac{x+y}{3\sqrt{(xy)}}$.

2. $\{1+2\sqrt{(-1)}+(-1)\}+\{1-2\sqrt{(-1)}+(-1)\}$
$=2\sqrt{(-1)}-2\sqrt{(-1)}=0.$

3. $\dfrac{2\sqrt{(xy)}+2y+2x-2\sqrt{(xy)}}{4\sqrt{(xy)}}=\dfrac{x+y}{2\sqrt{(xy)}}$.

4. $\{1+2\sqrt{(-1)}-1\}-\{1-2\sqrt{(-1)}-1\}=4\sqrt{(-1)}=\sqrt{(-16)}$.

5.
$$\begin{array}{l} x^2+\sqrt{2}.ax+a^2)\,x^4+a^4\,(x^2-\sqrt{2}.ax+a^2 \\ \qquad\qquad\qquad\; x^4+\sqrt{2}.ax^3+a^2x^2 \\ \qquad\qquad\qquad\; \hline \qquad\qquad\qquad\quad -\sqrt{2}.ax^3-a^2x^2+a^4 \\ \qquad\qquad\qquad\quad -\sqrt{2}.ax^3-2a^2x^2-\sqrt{2}.a^3x \\ \qquad\qquad\qquad\quad \hline \qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\ \qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\ \qquad\qquad\qquad\qquad\qquad \hline \end{array}$$

6. $m^2 - \sqrt{2}.mn + n^2) \, m^4 + n^4 \, (m^2 + \sqrt{2}.mn + n^2$

$$\begin{array}{r}
\underline{m^4 - \sqrt{2}.m^3n + m^2n^2} \\
\sqrt{2}.m^3n - m^2n^2 + n^4 \\
\underline{\sqrt{2}.m^3n - 2m^2n^2 + \sqrt{2}.mn^3} \\
m^2n^2 - \sqrt{2}.mn^3 + n^4 \\
\underline{m^2n^2 - \sqrt{2}.mn^3 + n^4}
\end{array}$$

7. $\sqrt{x}.\sqrt{}.(x^2 + 2xy + y^2) + \sqrt{x}.\sqrt{(x^2 - 2xy + y^2)}$
$= \sqrt{x}.(x + y) + \sqrt{x}.(x - y) = 2x\sqrt{x}.$

8. $$\frac{a\sqrt{a} + a\sqrt{b} - b\sqrt{a} - b\sqrt{b} - (a\sqrt{a} + b\sqrt{a} - a\sqrt{b} - b\sqrt{b})}{a - b}$$

$$= \frac{2a\sqrt{b} - 2b\sqrt{a}}{a - b}.$$

9. $a^2.\frac{c}{b} - 2a.\sqrt{\frac{c}{b}}.\sqrt{(cd)} + cd = \frac{a^2c}{b} - 2ac\sqrt{\frac{d}{b}} + cd.$

10. $a^{2\sqrt{2}} - 2a^{\sqrt{2}}.\frac{1}{a^{\sqrt{2}}} + \frac{1}{a^{2\sqrt{2}}} = a^{2\sqrt{2}} - 2 + \frac{1}{a^{2\sqrt{2}}}.$

11. $$\frac{x^2 + a^2 + 2\sqrt{(x^4 - a^4)} + x^2 - a^2 + x^2 + a^2 - 2\sqrt{(x^4 - a^4)} + x^2 - a^2}{(x^2 + a^2) - (x^2 - a^2)}$$

$$= \frac{4x^2}{2a^2} = \frac{2x^2}{a^2}.$$

12. $$\left\{ \frac{\sqrt{(1 - x^2)} + 1}{\sqrt{(1 + x)}} \right\} \div \left\{ \frac{\sqrt{(1 - x^2)} + 1}{\sqrt{(1 - x^2)}} \right\} = \frac{\sqrt{(1 - x^2)}}{\sqrt{(1 + x)}} = \sqrt{(1 - x)}.$$

13. $$\frac{x - 1}{x + 1} \left\{ \frac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{\sqrt{x} - 1} + \frac{(1 + \sqrt{x})(1 - \sqrt{x})}{\sqrt{x}(1 + \sqrt{x})} \right\}$$

$$= \frac{x - 1}{x + 1} \left\{ \sqrt{x} + 1 + \frac{1 - \sqrt{x}}{\sqrt{x}} \right\} = \frac{x - 1}{x + 1} \left(\frac{x + \sqrt{x} + 1 - \sqrt{x}}{\sqrt{x}} \right) = \frac{x - 1}{\sqrt{x}}.$$

14. $\frac{x}{4} + 3 - 2\sqrt{\left(\frac{x^2}{16} - 9\right)} + \frac{x}{4} - 3 = \frac{x}{2} - 2\sqrt{\left(\frac{x^2}{16} - 9\right)}.$

7.
$$\begin{array}{ll}
a\sqrt{(-a)}+b\sqrt{(-b)} & \\
a\sqrt{(-a)}-b\sqrt{(-b)} & \\
\hline
-a^3+ab\sqrt{(ab)} & \\
\quad -ab\sqrt{(ab)}+b^3 & \\
\hline
-a^3 \qquad\qquad +b^3 &
\end{array}$$

8.
$$\begin{array}{l}
a+\beta\sqrt{(-1)} \\
a-\beta\sqrt{(-1)} \\
\hline
a^2+a\beta\sqrt{(-1)} \\
\quad -a\beta\sqrt{(-1)}+\beta^2 \\
\hline
a^2 \qquad\qquad +\beta^2
\end{array}$$

9.
$$\begin{array}{l}
1-\sqrt{(1-e^2)} \\
1+\sqrt{(1-e^2)} \\
\hline
1-\sqrt{(1-e^2)} \\
\quad +\sqrt{(1-e^2)}-(1-e^2) \\
\hline
e^2
\end{array}$$

10.
$$\begin{array}{l}
e^{p\sqrt{(-1)}}+e^{-p\sqrt{(-1)}} \\
e^{p\sqrt{(-1)}}-e^{-p\sqrt{(-1)}} \\
\hline
e^{2p\sqrt{(-1)}}+1 \\
\qquad\qquad -1-e^{-2p\sqrt{(-1)}} \\
\hline
e^{2p\sqrt{(-1)}} \qquad -e^{-2p\sqrt{(-1)}}
\end{array}$$

CXVII.

1. $\dfrac{3x+3\sqrt{(xy)}-3\sqrt{(xy)}+3y}{9\sqrt{(xy)}}=\dfrac{x+y}{3\sqrt{(xy)}}$.

2. $\{1+2\sqrt{(-1)}+(-1)\}+\{1-2\sqrt{(-1)}+(-1)\}$
$=2\sqrt{(-1)}-2\sqrt{(-1)}=0.$

3. $\dfrac{2\sqrt{(xy)}+2y+2x-2\sqrt{(xy)}}{4\sqrt{(xy)}}=\dfrac{x+y}{2\sqrt{(xy)}}$.

4. $\{1+2\sqrt{(-1)}-1\}-\{1-2\sqrt{(-1)}-1\}=4\sqrt{(-1)}=\sqrt{(-16)}$.

5.
$$\begin{array}{l}
x^2+\sqrt{2}.ax+a^2)\,x^4+a^4\,(x^2-\sqrt{2}.ax+a^2 \\
\qquad\qquad\qquad x^4+\sqrt{2}.ax^3+a^2x^2 \\
\hline
\qquad\qquad\qquad -\sqrt{2}.ax^3-a^2x^2+a^4 \\
\qquad\qquad\qquad -\sqrt{2}.ax^3-2a^2x^2-\sqrt{2}.a^3x \\
\hline
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\
\qquad\qquad\qquad\qquad\qquad a^2x^2+\sqrt{2}.a^3x+a^4 \\
\hdashline
\end{array}$$

6.
$$\begin{array}{l}
m^2-\sqrt{2}.mn+n^2)\,m^4+n^4\,(m^2+\sqrt{2}.mn+n^2 \\
\qquad\qquad\qquad\underline{m^4-\sqrt{2}.m^3n+m^2n^2} \\
\qquad\qquad\qquad\qquad\sqrt{2}.m^3n-m^2n^2+n^4 \\
\qquad\qquad\qquad\qquad\underline{\sqrt{2}.m^3n-2m^2n^2+\sqrt{2}.mn^3} \\
\qquad\qquad\qquad\qquad\qquad\qquad m^2n^2-\sqrt{2}.mn^3+n^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{m^2n^2-\sqrt{2}.mn^3+n^4}
\end{array}$$

7. $\sqrt{x}.\sqrt{}.(x^2+2xy+y^2)+\sqrt{x}.\sqrt{(x^2-2xy+y^2)}$

$=\sqrt{x}.(x+y)+\sqrt{x}.(x-y)=2x\sqrt{x}.$

8. $$\frac{a\sqrt{a}+a\sqrt{b}-b\sqrt{a}-b\sqrt{b}-(a\sqrt{a}+b\sqrt{a}-a\sqrt{b}-b\sqrt{b})}{a-b}$$

$$=\frac{2a\sqrt{b}-2b\sqrt{a}}{a-b}.$$

9. $a^2.\frac{c}{b}-2a.\sqrt{\frac{c}{b}}.\sqrt{(cd)}+cd=\frac{a^2c}{b}-2ac\sqrt{\frac{d}{b}}+cd.$

10. $a^{2\sqrt{2}}-2a^{\sqrt{2}}.\frac{1}{a^{\sqrt{2}}}+\frac{1}{a^{2\sqrt{2}}}=a^{2\sqrt{2}}-2+\frac{1}{a^{2\sqrt{2}}}.$

11. $$\frac{x^2+a^2+2\sqrt{(x^4-a^4)}+x^2-a^2+x^2+a^2-2\sqrt{(x^4-a^4)}+x^2-a^2}{(x^2+a^2)-(x^2-a^2)}$$

$$=\frac{4x^2}{2a^2}=\frac{2x^2}{a^2}.$$

12. $\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}}\right\}\div\left\{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x^2)}}\right\}=\frac{\sqrt{(1-x^2)}}{\sqrt{(1+x)}}=\sqrt{(1-x)}.$

13. $$\frac{x-1}{x+1}\left\{\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}+\frac{(1+\sqrt{x})(1-\sqrt{x})}{\sqrt{x}(1+\sqrt{x})}\right\}$$

$$=\frac{x-1}{x+1}\left\{\sqrt{x}+1+\frac{1-\sqrt{x}}{\sqrt{x}}\right\}=\frac{x-1}{x+1}\left(\frac{x+\sqrt{x}+1-\sqrt{x}}{\sqrt{x}}\right)=\frac{x-1}{\sqrt{x}}.$$

14. $\frac{x}{4}+3-2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{x}{2}-2\sqrt{\left(\frac{x^2}{16}-9\right)}.$

15. $x+a-2\sqrt{(x^2-a^2)}+x-a=2x-2\sqrt{(x^2-a^2)}$.

16. $\sqrt[m]{(a^{2m-n+n}.b^{5m+1+m-1}.c^{3p+m-3p})}=\sqrt[m]{(a^{2m}.b^{6m}.c^m)}=a^2b^6c$.

17. $\{-1-a\sqrt{(-1)}\}^2=1+2a\sqrt{(-1)}-a^2$

$$\begin{aligned}\{1+2a\sqrt{(-1)}-a^2\}^2&=1-4a^2+a^4+4a\sqrt{(-1)}-2a^2-4a^3\sqrt{(-1)}\\ &\quad 1-6a^2+a^4+4a\sqrt{(-1)}-4a^3\sqrt{(-1)}\\ &\quad -1-a\sqrt{(-1)}\\ \hline &\quad -1+6a^2-a^4-4a\sqrt{(-1)}+4a^3\sqrt{(-1)}\\ &\qquad -a\sqrt{(-1)}+6a^3\sqrt{(-1)}-a^5\sqrt{(-1)}+4a^2-4a^4\\ \hline &\quad -1+10a^2-5a^4+(10a^3-a^5-5a)\sqrt{(-1)}\end{aligned}$$

18. $3\sqrt[3]{3}-(-8)+4\sqrt[3]{3}=8+7\sqrt[3]{3}$.

19. $\dfrac{6c^2}{x-1}.\sqrt{\left(\dfrac{x(4x^2-8x+4)}{3c^3}\right)}=\dfrac{6c^2}{x-1}\cdot\dfrac{2x-2}{c}.\sqrt{\dfrac{x}{3c}}=\dfrac{12c\sqrt{x}}{\sqrt{(3c)}}=4\sqrt{(3cx)}$.

20. $\dfrac{x}{x-7}\{\sqrt[3]{[3p^2\{x^3-21x^2+147x-343\}]}\}=\dfrac{x}{x-7}\cdot\sqrt[3]{(3p^2)}.(x-7)$

$=x\sqrt[3]{(3p^2)}$.

21. $2(n-1)\sqrt[3]{\left[\dfrac{1}{2n}.\left(-\dfrac{1}{n^3-3n^2+3n-1}\right)\right]}=2(n-1)\sqrt[3]{\left(\dfrac{1}{2n}\right)}.\left(-\dfrac{1}{n-1}\right)$

$=-2\sqrt[3]{\dfrac{1}{2n}}=-\sqrt[3]{\dfrac{8}{2n}}=-\sqrt[3]{\dfrac{4n^2}{n^3}}=\dfrac{1}{n}\sqrt[3]{(-4n^2)}$.

22. $2(n-1).3\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{2(n-1)\times 5}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=6n\sqrt{7}-6\sqrt{7}+\dfrac{4}{3}\sqrt{7}-2\sqrt{7}+\dfrac{10n}{3}\sqrt{7}-\dfrac{10}{3}\sqrt{7}-\dfrac{n}{3}\sqrt{7}$

$=9n\sqrt{7}-10\sqrt{7}$.

23. $\sqrt{(17^2-33)}\cdot\sqrt[3]{(65^2\cdot\cdot 129)}=\sqrt{256}-\sqrt[3]{4096}=16-16=0$.

CXVIII.

Adopt the notation of Art. 316; then

1. $\left.\begin{array}{r} x+y=10 \\ xy=21 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 100 \\ 4xy &= 84 \end{array}\right\}$; $x^2-2xy+y^2=16$; $x-y=4$, etc.

2. $\left.\begin{array}{r} x+y=16 \\ xy=55 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 256 \\ 4xy &= 220 \end{array}\right\}$ $x^2-2xy+y^2=36$; $x-y=6$, etc.

3. $\left.\begin{array}{r} x+y=9 \\ xy=14 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 81 \\ 4xy &= 56 \end{array}\right\}$; $x^2-2xy+y^2=25$; $x-y=5$, etc.

4. $\left.\begin{array}{r} x+y=94 \\ xy=2205 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 8836 \\ 4xy &= 8820 \end{array}\right\}$; $x^2-2xy+y^2=16$;
$x-y=4$, etc.

5. $\left.\begin{array}{r} x+y=13 \\ xy=30 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 169 \\ 4xy &= 120 \end{array}\right\}$; $x^2-2xy+y^2=49$; $x-y=7$, etc.

6. $\left.\begin{array}{r} x+y=38 \\ xy=360 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 1444 \\ 4xy &= 1440 \end{array}\right\}$; $x^2-2xy+y^2=4$; $x-y=2$, etc.

7. $\left.\begin{array}{r} x+y=14 \\ xy=24 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 196 \\ 4xy &= 96 \end{array}\right\}$; $x^2-2xy+y^2=100$;
$x-y=10$, etc.

8. $\left.\begin{array}{r} x+y=103 \\ xy=396 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 10609 \\ 4xy &= 1584 \end{array}\right\}$; $x^2-2xy+y^2=9025$;
$x-y=95$, etc.

9. $\left.\begin{array}{r} x+y=75 \\ xy=756 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 5625 \\ 4xy &= 3024 \end{array}\right\}$; $x^2-2xy+y^2=2601$;
$x-y=51$, etc.

10. $\left.\begin{array}{r} x+y=87 \\ xy=1512 \end{array}\right\}$; $\left.\begin{array}{rl} x^2+2xy+y^2 &= 7569 \\ 4xy &= 6048 \end{array}\right\}$; $x^2-2xy+y^2=1521$;
$x-y=39$, etc.

11. $\left.\begin{aligned} x+y&=\frac{7}{2}\\ xy&=\frac{10}{4}\end{aligned}\right\};\quad \left.\begin{aligned} x^2+2xy+y^2&=\frac{49}{4}\\ 4xy\quad&=\frac{40}{4}\end{aligned}\right\};\quad x^2-2xy+y^2=\frac{9}{4};$

$x-y=\frac{3}{2}$, etc.

12. $\left.\begin{aligned} x+y&=57\\ xy&=540\end{aligned}\right\}\quad \left.\begin{aligned} x^2+2xy+y^2&=3249\\ 4xy\quad&=2160\end{aligned}\right\};\quad x^2-2xy+y^2=1089;$

$x-y=33$, etc.

CXIX.

1. $x=7^2=49$.
2. $x=9^2=81$.
3. $x=5^2=25$.
4. $x=2^3=8$.
5. $x=3^3=27$.
6. $x=4^4=256$.
7. $x+9=36$, etc.
8. $x-7=49$, etc.
9. $x-15=64$, etc.
10. $x-9=144$, etc.
11. $4x-16=8$, etc.
12. $3\sqrt{x}=18$; $x=36$.
13. $\sqrt[3]{(2x+3)}=3$; $2x+3=27$; $2x=24$; $x=12$.
14. $c\sqrt{x}=a-b$; $c^2x=(a-b)^2$; $x=\frac{(a-b)^2}{c^2}$.
15. $\sqrt{(x^2-9)}=9-x$; $x^2-9=81-18x+x^2$; $18x=90$; $x=5$.
16. $x^2-11=x^2-2x+1$; $2x=12$; $x=6$.
17. $4x^2+5x-2=4x^2+4x+1$; $x=3$.
18. $9x^2-12x-51=9x^2-18x+9$; $6x=60$; $x=10$.
19. $x^2-ax+b=x^2+2ax+a^2$; $3ax=b-a^2$, etc.
20. $25x^2-3mx+n=m^2+10mx+25x^2$; $13mx=n-m^2$, etc.

CXX.

1. $\sqrt{(16+x)}=8-\sqrt{x}$; $16+x=64-16\sqrt{x}+x$; $16\sqrt{x}=48$; $\sqrt{x}=3$; $x=9$.

2. $x-16=64-16\sqrt{x}+x$; $16\sqrt{x}=80$; $\sqrt{x}=5$; $x=25$.

3. $\sqrt{(x+15)}=15-\sqrt{x}$; $x+15=225-30\sqrt{x}+x$; $30\sqrt{x}=210$;
$\sqrt{x}=7$; $x=49$.

4. $x-21=x-2\sqrt{x}+1$; $2\sqrt{x}=22$; $\sqrt{x}=11$; $x=121$.

5. $x-1=x+4-6\sqrt{(x+4)}+9$; $6\sqrt{(x+4)}=14$; $\sqrt{(x+4)}=\frac{7}{3}$;
$x+4=\frac{49}{9}$; $x=\frac{13}{9}$.

6. $1+2\sqrt{(3x+1)}+3x+1=4x+4$; $2\sqrt{(3x+1)}=x+2$;
Squaring both sides, $4(3x+1)=x^2+4x+4$; $x^2=8x$, etc.

7. $1-2\sqrt{(1-3x)}+1-3x=4(1-x)$; $2\sqrt{(1-3x)}=x-2$;
$4(1-3x)=x^2-2x+4$; $x^2+8x=0$, etc.

8. $a-\sqrt{x}=\sqrt{(x-a)}$; $a^2-2a\sqrt{x}+x=x-a$; $2\sqrt{x}=1+a$;
$4x=(1+a)^2$, etc.

9. $\sqrt{(x-m)}=\frac{m-2\sqrt{x}}{2}$; $x-m=\frac{m^2-4m\sqrt{x}+4x}{4}$; $-4m=m^2-4m\sqrt{x}$;
$\sqrt{x}=\frac{m+4}{4}$; $x=\left(\frac{m+4}{4}\right)^2$.

10. $\sqrt{(x-4)}=3-\sqrt{(x-1)}$; $x-4=9-6\sqrt{(x-1)}+x-1$;
$6\sqrt{(x-1)}=12$; $\sqrt{(x-1)}=2$; $x-1=4$; $x=5$.

CXXI.

1. $\sqrt{(x^2-9x)}+x-9=36$; $\sqrt{(x^2-9x)}=45-x$;
$x^2-9x=2025-90x+x^2$; $81x=2025$; $x=25$.

2. $x+\sqrt{(x^2-21x)}=35$; $x^2-21x=1225-70x+x^2$; $49x=1225$;
$x=25$.

3. $x+7+\sqrt{(x^2+7x)}=28$; $\sqrt{(x^2+7x)}=21-x$;
$x^2+7x=441-42x+x^2$; $49x=441$; $x=9$.

4. $x-15+\sqrt{(x^2-15x)}=105$; $\sqrt{(x^2-15x)}=120-x$;
$x^2-15x=14400-240x+x^2$; $225x=14400$; $x=64$.

5. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-24x+x^2$;
$20x=144$, etc.

6. $\sqrt{(3ax+x^2)}+3a+x-9a=0$; $\sqrt{(3ax+x^2)}=6a-x$;
$3ax+x^2=36a^2-12ax+x^2$; $15ax=36a^2$, etc.

7. $b^2-ax=b^2-ax-ab+bx$; $bx=ab$; $x=a$.

8. $2+\sqrt{x}-x=\frac{4+\sqrt{x}}{2}$; $4+2\sqrt{x}-2x=4+\sqrt{x}$; $\sqrt{x}=2x$; $x=4x^2$;
whence $x=0$ or $\frac{1}{4}$.

9. $x+28\sqrt{x}+192=x+36\sqrt{x}+128$; $8\sqrt{x}=64$; $x=64$.

10. $x-6\sqrt{x}-16=x-10\sqrt{x}+24$; $4\sqrt{x}=40$; $x=100$.

CXXII.

1. $x-3\sqrt{x}=4$; $3\sqrt{x}=x-4$; $9x=x^2-8x+16$; $x^2-17x=-16$, etc.

2. $14\sqrt{x}=x+45$; $196x=x^2+90x+2205$; $x^2-106x=-2205$, etc.

3. $9(7+2x^3)=25.(4x-3)$; $63+18x^2=100x-75$;
$18x^2-100x=-138$, etc.

4. $6x-11=249-2x^2$; $2x^2+6x=260$; $x^2+3x=130$, etc.

5. $6-x=4-4\sqrt{(2x-1)}+2x-1$; $4\sqrt{(2x-1)}=3x-3$;
$16(2x-1)=9x^2-18x+9$; $9x^2-50x=-25$, etc.

6. $x+12=2\sqrt{(4-3x)}$; $x^2+24x+144=4(4-3x)$; $x^2+36x=-128$.

7. $2x+7+2\sqrt{(6x^2-15x-126)}+3x-18=7x+1$;
$\sqrt{(6x^2-15x-126)}=x+6$; $6x^2-15x-126=x^2+12x+36$, etc.

8. $4(204-5x)=400-40\sqrt{(3x-68)}+3x-68$;
$40\sqrt{(3x-68)}=23x-484$;
$1600(3x-68)-529x^2-22264x+234256$;
$529x^2-27064x=-343056$;
$$x^2-\frac{27064}{529}x+\frac{183115024}{(529)^2}=\frac{183115024-181476624}{(529)^2}=\frac{1638400}{(529)^2}$$
$$x-\frac{13532}{529}=\pm\frac{1280}{529}, \text{ etc.}$$

9. $(\sqrt{x}-4)(\sqrt{x}+4)=33$; $x-16=33$; $x=49$.

10. $(\sqrt{x}+11).(\sqrt{x}-11)=608$; $x-121=608$; $x=729$.

11. $\sqrt{(x^2+17x+60)}=12$; $x^2+17x+60=144$; $x^2+17x=-84$, etc.

12. $x+3+2\sqrt{(x^2+11x+24)}+x+8=25x$; $2\sqrt{(x^2+11x+24)}=23x-11$;
$4(x^2+11x+24)=529x^2-506x+121$; $525x^2-550x=-25$;
$x^2-\frac{22x}{21}=-\frac{1}{21}$, etc.

13. $\sqrt{(25+x)}=8-\sqrt{(25-x)}$; $25+x=64-16\sqrt{(25-x)}+25-x$;
$8\sqrt{(25-x)}=32-x$; $64(25-x)=1024-64x+x^2$; $x^2=576$;
$x=\pm 24$.

14. $x+4+2\sqrt{(2x^2+7x-4)}+2x-1=36$; $2\sqrt{(2x^2+7x-4)}=33-3x$,
$4(2x^2+7x-4)=1089-198x+9x^2$; $x^2-226x=-1105$, etc.

15. $\sqrt{(13x-1)}=5+\sqrt{(2x-1)}$; $13x-1=25+10\sqrt{(2x-1)}+2x-1$;
$11x-25=10\sqrt{(2x-1)}$; $121x^2-550x+625=100(2x-1)$;
$121x^2-750x=-725$; $x^2-\frac{750}{121}x+\frac{140625}{(121)^2}=\frac{140625-87725}{(121)^2}$, etc.

16. $7x+1-2\sqrt{(21x^2+10x+1)}+3x+1=4$; $\sqrt{(21x^2+10x+1)}=5x-1$;
$21x^2+10x+1=25x^2-10x+1$; $4x^2=20x$; $x=5$ or 0.

17. $\sqrt{(4+x)}=3-\sqrt{x}$; $4+x=9-6\sqrt{x}+x$; $\sqrt{x}=\frac{5}{6}$; $x=\frac{25}{36}$.

18. $x+\sqrt{(x^2+9975)}=525$; $\sqrt{(x^2+9975)}=525-x$;
$x^2+9975x=275625-1050x+x^2$; $11025x=275625$, etc.

19. $\frac{x}{4}+3+2\sqrt{\left(\frac{x^2}{16}-9\right)}+\frac{x}{4}-3=\frac{2x}{3}$; $2\sqrt{\left(\frac{x^2}{16}-9\right)}=\frac{x}{6}$;
$\frac{x^2}{16}-9=\frac{x^2}{144}$; $\frac{8x^2}{144}=9$; $x^2=9\times 18$; $x=\pm 9\sqrt{2}$.

20. $x^2-1+6\sqrt{(x^2-1)}=16$; $x^2-1+6\sqrt{(x^2-1)}+9=25$;
$\sqrt{(x^2-1)}+3=\pm 5$; $\sqrt{(x^2-1)}=2$ or -8.
Hence $x^2-1=4$ or 64; $x^2=5$ or 65, etc.

21. $x^2-2ax+a^2+2ab+b^2=x^2+a^2+b^2-2ax+2bx-2ab$; $4ab=2bx$;
$x=2a$.

22. $x^2+2ax+a^2+2ab+b^2=b^2+a^2+x^2-2ab-2bx+2ax$; $4ab=-2bx$;
$x=-2a$.

23. $x+4-2\sqrt{(x^2+4x)}+x=x+\frac{3}{2}$; $x+\frac{5}{2}=2\sqrt{(x^2+4x)}$;
$x^2+5x+\frac{25}{4}=4x^2+16x$; $3x^2+11x=\frac{25}{4}$; $x^2+\frac{11x}{3}+\frac{121}{36}=\frac{196}{36}$;
$x+\frac{11}{6}=\pm\frac{14}{6}$; $x=\frac{1}{2}$ or $-\frac{25}{6}$.

24. $\sqrt{x}+1=x+\frac{5}{4}$; $\sqrt{x}=x+\frac{1}{4}$; $x-\sqrt{x}=-\frac{1}{4}$; $x-\sqrt{x}+\frac{1}{4}=0$;
$\sqrt{x}-\frac{1}{2}=0$; $x=\frac{1}{4}$.

25. $\sqrt{(4+x)}=\sqrt{3}+\sqrt{x}$; $4+x=3+2\sqrt{(3x)}+x$; $2\sqrt{(3x)}=1$;
$12x=1$, etc.

26. $\sqrt{(x+4)}=9-\sqrt{(x+5)}$; $x+4=81-18\sqrt{(x+5)}+x+5$;
$18\sqrt{(x+5)}=82$; $9\sqrt{(x+5)}=41$; $81(x+5)=1681$, etc.

27. $\sqrt{(x^2-4x)}+x-4=8$; $\sqrt{(x^2-4x)}=12-x$; $x^2-4x=144-24x+x^2$;
$20x=144$, etc.

28. $x^2-21=\sqrt{(x^2-9)}$; $x^4-42x^2+441=x^2-9$; $x^4-43x^2=-450$;
$x^4-43x^2+\frac{1849}{4}=\frac{49}{4}$; $x^2-\frac{43}{2}=\pm\frac{7}{2}$; $x^2=25$ or 18, etc.

29. $\sqrt{(50+x)}=2+\sqrt{(50-x)}$; $50+x=4+4\sqrt{(50-x)}+50-x$;
$x-2=2\sqrt{(50-x)}$; $x^2-4x+4=4(50-x)$; $x^2=196$, etc.

30. $\sqrt{(2x+4)}-1=\sqrt{\left(\frac{x}{2}+6\right)}$; $2x+4-2\sqrt{(2x+4)}+1=\frac{x}{2}+6$;
$\frac{3x}{2}-1=2\sqrt{(2x+4)}$; $9x^2-12x+4=16(2x+4)$; $9x^2-44x=60$;
$x^2-\frac{44}{9}x+\frac{484}{81}=\frac{1024}{81}$, etc.

31. $3+x+\sqrt{(3x+x^2)}=6$; $\sqrt{(3x+x^2)}=3-x$; $3x+x^2=9-6x+x^2$;
$9x=9$; $x=1$.

32. $\sqrt{(x-1)}+\sqrt{(x+1)}=1$; $\sqrt{(x-1)}=1-\sqrt{(x+1)}$;
$x-1=1-2\sqrt{(x+1)}+x+1$; $2\sqrt{(x+1)}=3$; $4(x+1)=9$, etc.

33. $3x+\sqrt{(4x-x^2)}=6x-2\sqrt{(4x-x^2)}$; $3\sqrt{(4x-x^2)}=3x$; $4x-x^2=x^2$;
$2x^2=4x$, etc.

34. $\sqrt{\{a-\sqrt{(ax+x^2)}\}}=\sqrt{a}-\sqrt{x}$; $a-\sqrt{(ax+x^2)}=a-2\sqrt{(ax)}+x$;
$\sqrt{(ax+x^2)}=2\sqrt{(ax)}-x$; $ax+x^2=4ax-4x\sqrt{(ax)}+x^2$;
$4x\sqrt{(ax)}=3ax$; whence $x=0$, or $4\sqrt{(ax)}=3a$; $16ax=9a^2$;
$16x=9a$, etc.

CXXIII.

1. $x-2=0$, or, $x-5=0$; $\therefore x=2$ or 5.

2. $x-3=0$, or, $x+7=0$; $\therefore x=3$ or -7.

3. $x+9=0$, or, $x+2=0$; $\therefore x=-9$ or -2.

4. $x-5a=0$, or, $x-6b=0$; $\therefore x=5a$ or $6b$.

5. $2x+7=0$, or, $3x-5=0$; $\therefore x=-\frac{7}{2}$ or $\frac{5}{3}$.

6. $19x-227=0$, or, $14x+83=0$; $\therefore x=\frac{227}{19}$ or $-\frac{83}{14}$.

7. $5x-4m=0$, or, $6x-11n=0$; $\therefore x=\frac{4m}{5}$ or $\frac{11n}{6}$.

8. $x^2+5ax+6a^2=0$, or, $x^2-7ax+12a^2=0$;
$(x+2a)(x+3a)=0$, or, $(x-4a)(x-3a)=0$;
$\therefore x+2a=0$, or, $x+3a=0$, or, $x-4a=0$, or, $x-3a=0$, etc.

9. $(x+2)(x-2)(x-a)(x-a)=0$; $\therefore x=\pm 2$ or a.

10. $x.x.(x-5)=0$, $\therefore x=0$ or 5.

11. $acx-2a+b=0$, or, $bcx+3a-b=0$;
$\therefore acx=2a-b$, or, $bcx=b-3a$, etc.

12. $cx-d=0$, or, $cx-e=0$, etc.

CXXIV.

1. Let m be one of the roots of the first equation, and $\frac{1}{m}$ one of the roots of the second equation.
Then $am^2+bm+c=0$, and $c'm^2+b'm+a'=0$.
Multiply the first by a' and the second by c; then
$aa'm^2+a'bm+a'c=0$, and $cc'm^2+b'cm+a'c=0$.
Subtracting $(aa'-cc')m^2+(a'b-b'c)m=0$.
Whence $(aa'-cc')m=-(a'b-b'c)$ . . . . (1).
Again, multiply the first by c' and the second by a; then
$ac'm^2+bc'm+cc'=0$, and $ac'm^2+ab'm+aa'=0$.
Subtracting $(bc'-ab')m=cc'-aa'$;
or $aa'-cc'=-(ab'-bc')m$ . . . . (2).
Multiplying (1) and (2) together, we get
$(aa'-cc')^2=(ab'-bc')(a'b-b'c)$.

2. $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$;

$\therefore \alpha^2+2\alpha\beta+\beta^2=\frac{b^2}{a^2}$, and $2\alpha\beta=\frac{2c}{a}$

$\therefore \alpha^2+\beta^2=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$.

3. $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$.

$$\therefore\ ac\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=ac\left(x^2-\frac{\alpha^2+\beta^2}{\alpha\beta}x+1\right)$$
$$=ac\left(x^2-\frac{b^2-2ac}{ac}x+1\right)$$
$$=acx^2+(2ac-b^2)x+ac.$$

4. The roots are $\dfrac{-b+\sqrt{(b^2-4ac)}}{2a}$ and $\dfrac{-b-\sqrt{(b^2-4ac)}}{2a}$, and that these may be *equal*, we must have $b^2-4ac=0$, or $c=\dfrac{b^2}{4a}$.

Putting this for c in the expression ax^2+bx+c,

$$ax^2+bx+\frac{b^2}{4a}=\frac{4a^2x^2+4abx+b^2}{4a}=\left(\frac{2ax+b}{2\sqrt{a}}\right)^2.$$

5. $\alpha+\beta=1+a$, and $\alpha\beta=\dfrac{1+a+a^2}{2}$.

$\alpha^2+2\alpha\beta+\beta^2=1+2a+a^2$, and $2\alpha\beta=1+a+a^2$

$\therefore\ \alpha^2+\beta^2=a$.

CXXV.

1. $(x-5)(x-6)=0$, or, $x^2-11x+30=0$.

2. $(x-4)(x+5)=0$, or, $x^2+x-20=0$.

3. $(x+2)(x+7)=0$, or, $x^2+9x+14=0$.

4. $\left(x-\dfrac{1}{2}\right)\left(x-\dfrac{2}{3}\right)=0$, or, $(2x-1)(3x-2)=0$, or, $6x^2-7x+2=0$.

5. $(x-7)\left(x+\dfrac{5}{9}\right)=0$, or, $(x-7)(9x+5)=0$, or, $9x^2-58x-35=0$.

6. $(x-\sqrt{3})(x+\sqrt{3})=0$, or, $x^2-3=0$.

7. $(x-m-n)(x-m+n)=0$, or, $x^2-2mx+m^2-n^2=0$.

8. $\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)=0$, or, $x^2-\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0$.

9. $\left(x+\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$, or, $x^2+\frac{\alpha^2-\beta^2}{\alpha\beta}x-1=0$.

CXXVI.

1. One root is found by trial to be 2; then
$$x^3-11x^2+36x-36=(x-2)(x^2-9x+18)=(x-2)(x-3)(x-6).$$

2. One root is found by trial to be 1; then
$$x^3-7x^2+14x-8=(x-1)(x^2-6x+8)=(x-1)(x-2)(x-4).$$

3. One root is found by trial to be -1; then
$$x^3-5x^2-46x-40=(x+1)(x^2-6x-40)=(x+1)(x-10)(x+4).$$

4. One root is found by trial to be -1; then
$$4x^3+6x^2+x-1=(x+1)(4x^2+2x-1)=(x+1)4.\left(x^2+\frac{x}{2}-\frac{1}{4}\right)$$
and since the roots of $x^2+\frac{x}{2}-\frac{1}{4}=0$ are $\frac{\sqrt{5}-1}{4}$ and $\frac{-\sqrt{5}-1}{4}$,
$$x^2+\frac{x}{2}-\frac{1}{4}=\left(x+\frac{1-\sqrt{5}}{4}\right).\left(x+\frac{1+\sqrt{5}}{4}\right),$$ and therefore
$$4x^3+6x^2+x-1=4(x+1)\left(x+\frac{1-\sqrt{5}}{4}\right)\left(x+\frac{1+\sqrt{5}}{4}\right).$$

5. One root is found by trial to be -1; then
$$6x^3+11x^2-9x-14=(x+1)(6x^2+5x-14)=(x+1)(x+2)(6x-7).$$

6. If we put $x=-y-z$, the expression becomes zero, which shows that $x+y+z$ is one of its factors; the other is found by division to be $x^2+y^2+z^2-xy-xz-yz$.

7. If we put $a=b+c$, the expression becomes zero, which shows that $a-b-c$ is one of its factors; the other is found by division to be $a^2+b^2+c^2+ab+ac-bc$.

8. One root is found by trial to be 1; then

$$3x^3 - x^2 - 23x + 21 = (x-1)(3x^2+2x-21) = (x-1)(3x-7)(x+3).$$

9. One root is found by trial to be 1; then

$$2x^3 - 5x^2 - 17x + 20 = (x-1)(2x^2-3x-20) = (x-1)(x-4)(2x+5).$$

10. One root is found by trial to be -1; then

$$15x^3 + 41x^2 + 5x - 21 = (x+1)(15x^2+26x-21)$$
$$= (x+1)(3x+7)(5x-3).$$

CXXVII.

1. $x^4 - 12x^2 = 13$; $x^4 - 12x^2 + 36 = 49$; $x^2 - 6 = \pm 7$, etc.

2. $x^6 + 14x^3 = -24$; $x^6 + 14x^3 + 49 = 25$; $x^3 + 7 = \pm 5$, etc.

3. $x^8 + 22x^4 = -21$; $x^8 + 22x^4 + 121 = 100$; $x^4 + 11 = \pm 10$, etc.

4. $x^{2m} + 3x^m + \frac{9}{4} = \frac{25}{4}$; $x^m + \frac{3}{2} = \pm\frac{5}{2}$, etc.

5. $x^{4n} - \frac{5}{3}x^{2n} + \frac{25}{36} = \frac{100}{36}$; $x^{2n} - \frac{5}{6} = \pm\frac{10}{6}$, etc.

6. $x - \frac{9}{2}x^{\frac{1}{2}} + \frac{81}{16} = \frac{121}{16}$; $x^{\frac{1}{2}} - \frac{9}{4} = \pm\frac{11}{4}$, etc.

7. $x^{-2} + 3x^{-1} + \frac{9}{4} = \frac{97}{36}$; $x^{-1} + \frac{3}{2} = \pm\frac{\sqrt{97}}{6}$; $x^{-1} = \frac{-9 \pm \sqrt{97}}{6}$

$$\therefore x = \frac{6}{-9 \pm \sqrt{97}}.$$

8. $x^{-2n} - x^{-n} + \frac{1}{4} = \frac{81}{4}$; $x^{-n} - \frac{1}{2} = \pm\frac{9}{2}$; $x^{-n} = 5$ or -4,

$$\therefore \frac{1}{x^n} = \frac{1}{5} \text{ or } -\frac{1}{4}; \; x = \left(\frac{1}{5}\right)^{\frac{1}{n}} \text{ or } \left(-\frac{1}{4}\right)^{\frac{1}{n}}.$$

9. $x^2 - 2x + 5 + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$;

$(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} + 9 = 25$; $(x^2 - 2x + 5)^{\frac{1}{2}} + 3 = \pm 5$;

$(x^2 - 2x + 5)^{\frac{1}{2}} = 2$ or -8; $x^2 - 2x + 5 = 4$ or 64, etc.

10. $2x^2-2x+10\sqrt{(2x^2-5x+6)}=3x+33$;
$2x^2-5x+6+10\sqrt{(2x^2-5x+6)}=39$;
$(2x^2-5x+6)+10\sqrt{(2x^2-5x+6)}+25=64$;
$\sqrt{(2x^2-5x+6)}+5=\pm 8$; $\sqrt{(2x^2-5x+6)}=3$ or -13;
$2x^2-5x+6=9$ or 169, etc.

11. $3x^2-6\sqrt{(3x^2-2ax+4)}+12=2ax+a^2+2a$;
$3x^2-2ax+4-6\sqrt{(3x^2-2ax+4)}=a^2+2a-8$;
$(3x^2-2ax+4)-6\sqrt{(3x^2-2ax+4)}+9=a^2+2a+1$;
$\sqrt{(3x^2-2ax+4)}-3=\pm(a+1)$;
$3x^2-2ax+4=(4+a)^2$ or $(2-a)^2$, etc.

12. $x^2-ax+a^2-2\sqrt{(x^2-ax+a^2)}=-2a+a^2$;
$(x^2-ax+a^2)-2\sqrt{(x^2-ax+a^2)}+1=a^2-2a+1$;
$\sqrt{(x^2-ax+a^2)}-1=\pm(a-1)$; $x^2-ax+a^2=a^2$ or $(2-a)^2$, etc.

CXXVIII.

1. $\frac{2}{3}, \frac{6}{7}, \frac{7}{9}; \frac{42}{63}, \frac{54}{63}, \frac{49}{63}$, etc.

2. $\frac{x+3y}{x+2y}, \frac{x+2y}{x+y}; \frac{x^2+4xy+3y^2}{(x+2y)(x+y)}, \frac{x^2+4xy+4y^2}{(x+2y)(x+y)}$, etc.

3. $\frac{x-5y}{x-4y}, \frac{x-3y}{x-2y}; \frac{x^2-7xy+10y^2}{(x-4y)(x-2y)}, \frac{x^2-7xy+12y^2}{(x-4y)(x-2y)}$, etc.

4. Let x be the number.

Then $\frac{a+x}{b+x}=\frac{c}{d}$; $ad+dx=bc+cx$;

$(c-d)x=ad-bc$; $x=\frac{ad-bc}{c-d}$.

5. Let $x:y$ be the ratio. Then

$$\left.\begin{aligned} x^2+y^2&=181 \\ xy&=90 \end{aligned}\right\}, \quad \left.\begin{aligned} x^2+y^2+2xy&=361 \\ x^2+y^2-2xy&=1 \end{aligned}\right\}, \quad \left.\begin{aligned} x+y&=19 \\ x-y&=\pm 1 \end{aligned}\right\}, \quad \begin{aligned} x&=10 \text{ or } 9 \\ y&=9 \text{ or } 10 \end{aligned}$$

CXXIX.

1. $x:y=6:9=2:3.$ 2. $x:y=b:a.$

3. $(a-c)x=(b+d)y$; $x:y=b+d:a-c.$

4. $x^2+2xy+y^2=6y^2$; $x+y=\pm\sqrt{6}.y$;

$x=(\pm\sqrt{6}-1)y$; $x:y=\pm\sqrt{6}-1:1.$

5. $x^2-12xy+36y^2=49y^2$; $x-6y=\pm 7y$;

$x=13y$ or $-y$; $x:y=13:1$, or, $-1:1.$

6. $x^2+mxy+\frac{m^2y^2}{4}=\frac{(m^2+4n^2)y^2}{4}$; $x+\frac{my}{2}=\frac{\pm\sqrt{(m^2+4n^2)}y}{2}$;

$x=\frac{-m\pm\sqrt{(m^2+4n^2)}}{2}y$, etc.

7. Let $3x$ and $4x$ be the numbers. Then

$\frac{3x+4x}{9x^2+16x^2}=\frac{7}{50}$; $7\times 25x^2=50\times 7x$; $x=2.$

Hence the numbers are 6 and 8.

8. Let $6x$ and $7x$ be the numbers. Then

$\frac{6x+12}{7x+12}=\frac{12}{13}$; $78x+156=84x+144$; $6x=12$; $x=2$, etc.

9. Let $7x$ and $13x$ be the numbers. Then $20x=100$; $x=5$, etc.

10. Let x and y be the numbers. Then $x^2-y^2=48$, and $\frac{x+y}{x-y}=\frac{12}{1}.$

Hence $x+y=12x-12y$, or, $13y=11x.$

Then $x^2-\frac{121x^2}{169}=48$; $48x^2=169\times 48$; $x=13$, etc.

11. Let x be the value of a gold coin, and y the value of a silver one. Then $5x+4y=3x+12y$; $2x=8y$; $x:y=4:1.$

12. Let x be the value of a gold coin, and y the value of a silver one. Then $8x+9y=6x+19y$; $2x=10y$; $y:x=1:5.$

CXXX.

1. $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$.

2. $\frac{3}{7}\times\frac{14}{9}\times\frac{4}{3}=\frac{8}{9}$.

3. $\frac{x^2-y^2}{x^3+y^3}\times\frac{x^2-xy+y^2}{x+y}=\frac{x-y}{x+y}$.

4. $\frac{a^2-b^2+2bc-c^2}{a^2-b^2-2bc-c^2}\times\frac{a+b+c}{a+b-c}=\frac{(a+b-c)(a-b+c)(a+b+c)}{(a+b+c)(a-b-c)(a+b-c)}$, etc.

5. $\frac{m^3+n^3}{m^3-n^3}\times\frac{m-n}{m+n}=\frac{m^2-mn+n^2}{m^2+mn+n^2}$.

6. $\frac{(x+2)(x+3)}{(y-3)(y-4)}\times\frac{y(y-3)}{x(x+3)}=\frac{(x+2)y}{(y-4)x}$.

CXXXI.

1. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$ and $c=\lambda d$.

Then $\frac{a+b}{a}=\frac{\lambda b+b}{\lambda b}=\frac{\lambda+1}{\lambda}$,

and $\frac{c+d}{c}=\frac{\lambda d+d}{\lambda d}=\frac{\lambda+1}{\lambda}$.

2. Let $\frac{a}{b}=\lambda$, then $\frac{c}{d}=\lambda$. Then $a=\lambda b$, $c=\lambda d$.

Then $\frac{a^2-b^2}{b^2}=\frac{\lambda^2b^2-b^2}{b^2}=\lambda^2-1$,

and $\frac{c^2-d^2}{d^2}=\frac{\lambda^2d^2-d^2}{d^2}=\lambda^2-1$.

3. Let $\frac{a_1}{b_1}=\lambda$, then $\frac{a_2}{b_2}=\lambda$. Then $a_1=\lambda b_1$, and $a_2=\lambda b_2$.

Then $\frac{m_1a_1+m_2a_2}{m_1b_1+m_2b_2}=\frac{m_1\lambda b_1+m_2\lambda b_2}{m_1b_1+m_2b_2}=\lambda=\frac{a_1}{b_1}$.

4. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{3a^2+ab+2b^2}{3a^2-2b^2}=\frac{3b^2\lambda^2+b^2\lambda+2b^2}{3b^2\lambda^2-2b^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}$$

$$\frac{3c^2+cd+2d^2}{3c^2-2d^2}=\frac{3d^2\lambda^2+d^2\lambda+2d^2}{3d^2\lambda^2-2d^2}=\frac{3\lambda^2+\lambda+2}{3\lambda^2-2}.$$

5. Let $a = b\lambda$ and $c = d\lambda$. Then

$$\frac{a^2+3ab+b^2}{c^2+3cd+d^2}=\frac{b^2\lambda^2+3b^2\lambda+b^2}{d^2\lambda^2+3d^2\lambda+d^2}=\frac{b^2(\lambda^2+3\lambda+1)}{d^2(\lambda^2+3\lambda+1)}=\frac{b^2}{d^2}$$

$$\frac{2ab+3b^2}{2cd+3d^2}=\frac{2b^2\lambda+3b^2}{2d^2\lambda+3d^2}=\frac{b^2(2\lambda+3)}{d^2(2\lambda+3)}=\frac{b^2}{d^2}.$$

6. Let $\frac{a}{b}=\lambda$; then $\frac{c}{d}=\lambda$ and $\frac{e}{f}=\lambda$.

Then $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\frac{mc-ne}{md-nf}=\frac{md\lambda-nf\lambda}{md-nf}=\lambda=\frac{a}{b}.$

7. $$\frac{a-\frac{ma}{n}}{b-\frac{mb}{n}}=\frac{a\left(1-\frac{m}{n}\right)}{b\left(1-\frac{m}{n}\right)}=\frac{a}{b}.$$

8. Let $a = b\lambda$, $c = d\lambda$, $e = f\lambda$.

Then $\frac{ac}{bd}=\frac{bd\lambda^2}{bd}=\lambda^2$

and $\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}=\frac{lb^2\lambda^2+md^2\lambda^2+nf^2\lambda^2}{lb^2+md^2+nf^2}=\lambda^2.$

9. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_3\lambda$.

Then $\frac{a_1^2+a_2^2+a_3^2}{b_1^2+b_2^2+b_3^2}=\frac{b_1^2\lambda^2+b_2^2\lambda^2+b_3^2\lambda^2}{b_1^2+b_2^2+b_3^2}=\lambda^2=\frac{a_1^2}{b_1^2}.$

10. Let $a_1 = b_1\lambda$; $a_2 = b_2\lambda$; $a_3 = b_3\lambda$.

Then $\frac{a_1a_2+a_2a_3+a_3a_1}{b_1b_2+b_2b_3+b_3b_1}=\frac{b_1b_2\lambda^2+b_2b_3\lambda^2+b_3b_1\lambda^2}{b_1b_2+b_2b_3+b_3b_1}=\lambda^2=\frac{a_1^2}{b_1^2}.$

11. $(a^2-ab+b^2)(c^2+cd+d^2)=(a^2+ab+b^2)(c^2-cd+d^2)$;
$(a^2+b^2)(c^2+cd+d^2-c^2+cd-d^2)=ab(c^2-cd+d^2+c^2+cd+d^2)$;
$(a^2+b^2)(2cd)=ab(2c^2+2d^2)$; $a^2cd+b^2cd=abc^2+abd^2$;
$a^2cd-abd^2=abc^2-b^2cd$; $ad(ac-bd)=bc(ac-bd)$;
$ad=bc$; and therefore $a:b=c:d$.

12. $(a^2+b^2)(c^2-d^2)=(a^2-b^2)(c^2+d^2)$;
$a^2c^2+b^2c^2-a^2d^2-b^2d^2=a^2c^2+a^2d^2-b^2c^2-b^2d^2$; $2b^2c^2=2a^2d^2$;
$bc=ad$; and $\therefore\ a:b=c:d$.

13. Let $\frac{a}{b}=\lambda$ and $\frac{c}{d}=\lambda$; then $a=b\lambda$ and $c=d\lambda$.

Then $\frac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\frac{(b\lambda+d\lambda)(b^2\lambda^2+d^2\lambda^2)}{(b\lambda-d\lambda)(b^2\lambda^2-d^2\lambda^2)}=\frac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}$.

14. Let $a_1=b_1\lambda$ and $a_2=b_2\lambda$.

Then $\frac{\sqrt{(a_1{}^2+a_2{}^2)}}{\sqrt{(b_1{}^2+b_2{}^2)}}=\sqrt{\frac{b_1{}^2\lambda^2+b_2{}^2\lambda^2}{b_1{}^2+b_2{}^2}}=\sqrt{(\lambda^2)}=\lambda=\frac{a_1}{b}$.

CXXXII.

1. $(a-b)c=(b-c)b$; $ac-bc=b^2-bc$; $ac=b^2$; $a:b=b:c$.

2. Let $a=\lambda b$ and $c=\lambda d$.

Then $\frac{(a^2+b^2)(a+b)}{a^3}=\frac{(\lambda^2b^2+b^2)(\lambda b+b)}{\lambda^3b^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$,

and $\frac{(c^2+d^2)(c+d)}{c^3}=\frac{(\lambda^2d^2+d^2)(\lambda d+d)}{\lambda^3d^3}=\frac{(\lambda^2+1)(\lambda+1)}{\lambda^3}$.

Also, $\frac{\sqrt[4]{(ma^4+nc^4)}}{\sqrt[4]{(mb^4+nd^4)}}=\sqrt[4]{\frac{mb^4\lambda^4+nd^4\lambda^4}{mb^4+nd^4}}=\sqrt[4]{\lambda^4}=\lambda=\frac{a}{b}$.

3. Let $a=b\lambda$ and $c=d\lambda$.

Then $\frac{ma-nb}{ma+nb}=\frac{mb\lambda-nb}{mb\lambda+nb}=\frac{m\lambda-n}{m\lambda+n}$,

and $\frac{mc-nd}{mc+nd}=\frac{md\lambda-nd}{md\lambda+nd}=\frac{m\lambda-n}{m\lambda+n}$.

4. $(5a+3b)(7b+3c)=(7a+3b)(5b+3c)$;
$35ab+15ac+21b^2+9bc=35ab+21ac+15b^2+9bc$; $6b^2=6ac$;
$b^2=ac$; $a:b=b:c$.

5. Let $a:b=c:d$, and suppose a to be the greatest of the four.
Then $\because \frac{a}{b}=\frac{c}{d}$, and a is greater than b, c is greater than d,
and $\because \frac{a}{c}=\frac{b}{d}$, and a is greater than c, b is greater than d.
Again, if $a+b:m+n=m-n:a-b$,
and $a+b$ is greater than $m+n$
$m-n$ is greater than $a-b$
$\therefore a+b+m-n$ is greater than $m+n+a-b$
$\therefore 2b$ is greater than $2n$; and $\therefore b$ is greater than n.

6. $(x-1)(x+2)=(x-2)(2x+1)$; $x^2+x-2=2x^2-3x-2$; $x^2=4x$;
$x=4$ or 0.

7. $\frac{a}{b}+1=\frac{c}{d}+1$; $\frac{a}{b}=\frac{c}{d}$.

8. Suppose the bicycle went $5x$ and the tricycle $4x$ yards per minute. Then the tricycle had a start of $2x$ yards out of 1760 yards. Also the bicycle went 880 yards while the tricycle went
$1760-2x-176$. $\therefore 1760:1760-2x-176=5:4$;
$1760\times 4=(1584-2x)\times 5$; $10x=7920-7040$, etc.

9. $\frac{a}{b}=\frac{c}{d}$; $\therefore \frac{a^2}{b^2}=\frac{c^2}{d^2}$; $\therefore \frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$;
$\therefore \frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}$; and since b^2 is greater than d^2,
$\therefore a^2-b^2$ is greater than c^2-d^2
$\therefore a^2+d^2$ is greater than b^2+c^2.

10. $(10a+b)(12c+d)=(10c+d)(12a+b)$;
$120ac+12bc+10ad+bd=120ac+12ad+10bc+bd$; $2bc=2ad$
$bc=ad$; $a:b=c:d$.

11. $2x=3y$, and $xy=600$.

Hence $x\times\dfrac{2x}{3}=600$; $x^2=900$; $x=30$, etc.

12. Let $\dfrac{a}{b}=\lambda$; then $\dfrac{b}{c}=\lambda$.

Then (1.) $\dfrac{a}{a+b}=\dfrac{\lambda b}{\lambda b+b}=\dfrac{\lambda}{\lambda+1}$

$$\frac{a-b}{a-c}=\frac{\lambda b-b}{\lambda b-c}=\frac{\lambda^2c-\lambda c}{\lambda^2c-c}=\frac{\lambda^2-\lambda}{\lambda^2-1}=\frac{\lambda}{\lambda+1}.$$

(2.) $(a^2+b^2)(b^2+c^2)=(\lambda^2b^2+b^2)(\lambda^2c^2+c^2)$

$$=\lambda^2c^2(\lambda^2+1)(\lambda^2c^2+c^2)=\lambda^2c^4(\lambda^2+1)^2$$

$$(ab+bc)^2=(\lambda b^2+\lambda c^2)^2=(\lambda^3c^2+\lambda c^2)^2=\lambda^2c^4(\lambda^2+1)^2.$$

13. Let $a=b\lambda$ and $c=d\lambda$.

Then $\dfrac{a+b}{b}=\dfrac{b\lambda+b}{b}=\lambda+1$;

and $\dfrac{c+d}{d}=\dfrac{d\lambda+d}{d}=\lambda+1$.

Hence $\dfrac{ab-bc-dx+(bc+dx)}{bc+dx}=\dfrac{a-b-c+(b+c)}{b+c}$;

or, $\dfrac{ab}{bc+dx}=\dfrac{a}{b+c}$; $\dfrac{b}{bc+dx}=\dfrac{1}{b+c}$; $b^2+bc=bc+dx$, etc.

14. Let $a=\lambda b$, and $b=\lambda c$.

Then $\dfrac{a+mb}{a-mb}=\dfrac{\lambda b+mb}{\lambda b-mb}=\dfrac{\lambda+m}{\lambda-m}$;

and $\dfrac{b+mc}{b-mc}=\dfrac{\lambda c+mc}{\lambda c-mc}=\dfrac{\lambda+m}{\lambda-m}$.

15. $a=\dfrac{5b}{4}$; and therefore $\dfrac{a^2-b^2}{a^2+b^2}=\dfrac{\dfrac{25b^2}{16}-b^2}{\dfrac{25b^2}{16}+b^2}=\dfrac{9}{41}$.

16. Let the sides be $2\frac{1}{2}x$, $3\frac{3}{4}x$, and $4x$ yards.

Then $2\frac{1}{2}x+3\frac{3}{4}x+4x=205$; $\dfrac{41}{4}x=205$; $x=20$, etc.

17. Let the sides be $3x$, $4x$ and $5x$ yards.
Then $3x+4x+5x=480$; $12x=480$; $x=40$, etc.

18. Let $a+b$ be the greatest term of the proportion, then $a-b$ is the least.
Also, $\frac{a+b}{p+q}=\frac{p-q}{a-b}$.
Hence $\frac{(a+b)-(p+q)}{p+q}=\frac{(p-q)-(a-b)}{a-b}$.
Now $p+q$ is greater than $a-b$.
$\therefore$ $(a+b)-(p+q)$ is greater than $(p-q)-(a-b)$.
And $\therefore$ $(a+b)+(a-b)$ is greater than $(p+q)+(p-q)$.

19. Let x be the rate of the man's rowing, y the rate of the stream, in miles per hour.
Then $x+y=$ the man's rate down stream,
and $x-y=$ the man's rate up stream.
Hence $x+y:x-y=5:3$; $3x+3y=5x-5y$; $4y=x$.
Also $\frac{30}{x+y}+\frac{30}{x-y}=12$; $\frac{30}{5y}+\frac{30}{3y}=12$; $16=12y$; $y=1\frac{1}{3}$.

20. Let C contain x pints of brandy and y pints of water.
Then A contains $x+y$ pints of water, and B $x+y$ pints of brandy.
Hence when B and C are mixed, the mixture contains $2x+y$ pints of brandy;
and when A and C are mixed, the mixture contains x pints of brandy;
therefore $2x+y:x=9:1$; $2x+y=9x$; $y=7x$, etc.

21. Let x be the number of quarters; y the price of each in shillings.
Then selling price: $xy=105:100$
$\therefore$ $\frac{105xy}{100}=xy+16\times 20$; $\therefore$ $xy=6400$.
Again $x(y+5)=xy+20y$; $\therefore$ $x=4y$.
Hence $4y^2=6400$; $y^2=1600$; $y=40$; $x=160$.

22. Let x be the price of the horse in pounds.
Then since cost price : gain $=100$: gain per cent.
$\therefore x:144-x=100:x$
$\therefore x^2+100x=14400$; $x^2+100x+2500=16900$; $x=80$.

23. Let x be the cost of the goods in pounds.
Then $x:96-x=100:x$; $x^2+100x=9600$; $x=60$.

24. Let x be the cost of the sheep in pounds.
Then $x:24-x=100:x$; $x^2+100x=2400$; $x=20$.

25. Let the first crew row x yards, and the second y yards at each stroke.
Then in 8 minutes the first crew row $320x$ yards,
and in 8 minutes the second crew row $360y$ yards.
Also the second crew has $4x$ yards start.
Hence $320x=360y+4x$; $316x=360y$; $79x=90y$, etc.

26. Let x be the rate of the fast train, y the rate of the slow train.

Then $\frac{180}{y}+\frac{30}{y}=$ time for journey by slow train

$\frac{180}{x}+\frac{15}{y}=$ time for journey by fast train

$$\therefore \frac{210}{y} : \frac{180y+15x}{xy}=14:9; \text{ or, } \frac{14}{y} : \frac{12y+x}{xy}=14:9.$$

Hence $14\times 9\times x=(12y+x)\times 14$; $8x=12y$; $2x=3y$.
Again $x=y+15$; $\therefore 2y+30=3y$; $y=30$; $x=45$.

27. Let x be the worth of the article, y the selling price, in pounds.
Then $x:x-y=100:x$
$\therefore x^2=100x-100y$; $x^2-100x+2500=2500-100y$.
To obtain a real value for x, $100y$ must not be greater than 2500;
$\therefore y$ cannot be greater than 25.

CXXXIII.

1. Let $A=\frac{m}{B}$ and $B=\frac{n}{C}$.

Then $A=m\div\frac{n}{C}=\frac{m}{n}\cdot C$; $\therefore A\propto C$.

2. Let $A=mB$. Then $\frac{A}{P}=m\cdot\frac{B}{P}$; $\therefore \frac{A}{P}\propto\frac{B}{P}$.

3. Let $A=mB$ and $C=nD$. Then $AC=mn.BD$, etc.

4. $5:7=12:x$; $5x=84$, etc.

5. $x:\frac{1}{y}=10:\frac{1}{2}$; $\frac{x}{2}=\frac{10}{y}$; $\frac{4}{2}=\frac{10}{y}$; $y=5$.

6. $x:yz=1:2\times3$; $6x=yz$; $6\times4=y\times2$; $y=12$.

7. $x:\frac{y}{z}=6:\frac{4}{3}$; $\frac{4x}{3}=\frac{6y}{z}$; $\frac{4x}{3}=\frac{6\times5}{7}$; $x=3\frac{3}{14}$.

8. $3x+5y:5x+3y=31:25$; $75x+125y=155x+93y$; $32y=80x$, etc.

9. Let $A=mB$, and $B^3=nC^2$; then $\frac{A^3}{m^3}=nC^2$; $A^3=\frac{n}{m^3}\cdot C^2$;

   $A=\frac{n^{\frac{1}{3}}}{m}\cdot C^{\frac{2}{3}}$; $\therefore A\propto C^{\frac{2}{3}}$.

10. $z:xy=4:2$; $30:3x=2:1$; $x=5$.

11. $A:B=8:12$; $12A=8B$; $A=\frac{2}{3}B$.

12. $x^2:y^3=9:64$; $64x^2=9y^3$.

13. $x^2:\frac{1}{y^3}=4:\frac{1}{27}$; $\frac{x^2}{27}=\frac{4}{y^3}$; $x^2=\frac{108}{y^3}$.

14. $x^3:y^2=27:4$; $4x^3=27y^2$.

15. Let $x=mz$, and $y=\frac{n}{z}$; then $xy=mn$; $x=mn.\frac{1}{y}$.

16. The area of a triangle $=\frac{1}{2}$(base × altitude).
    Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the triangles; then $a_1b_1=a_2b_2$, or, $a_1:a_2=b_2:b_1$.

17. The area of a parallelogram = base × altitude.

Let a_1, a_2 be the altitudes; b_1, b_2 the bases of the parallelograms.

Then $a_1b_1=a_2b_2$; or, $a_1:a_2=b_2:b_1$.

18. Let $y=p+mx+nx^2$.

$$\text{Then } \left.\begin{aligned}6&=p+\ m+\ n\\ 11&=p+2m+4n\\ 18&=p+3m+9n\end{aligned}\right\};\quad \left.\begin{aligned}5&=m+3n\\ 7&=m+5n\end{aligned}\right\};\ n=1,\ m=2,\ p=3.$$

$$\therefore\ y=3+2x+x^2.$$

19. $10\times27:9\times9\times10=2\times27:3\times3\times$ required height in feet.

$$\text{Height}=\frac{9\times9\times10\times2\times27}{10\times27\times3\times3}\text{ ft.}=18\text{ ft.}$$

20. Let g, n, l, b represent the area of glass, the number, length, and breadth of the panes respectively.

Then suppose $g=m.nlb$, where m is a constant,

$$n=\frac{p}{b^2}, \text{ where } p \text{ is a constant.}$$

$$l=\frac{q}{b}, \text{ where } q \text{ is a constant.}$$

$$\text{Then}\qquad g=m.\frac{p}{b^2}.l.\frac{q}{l}=mpq.\frac{1}{b^2}=mpq.\frac{l^2}{q^2}=\frac{mp}{q}.l^2$$

$\therefore$ g varies as l^2.

CXXXIV.

1. $a=2,\ d=3,\ n-1=16$

$$z=2+16\times3=2+48=50.$$

2. $a=4,\ d=4,\ n-1=49$

$$z=4+4\times49=4+196=200.$$

3. $a=7,\ d=\frac{1}{4},\ n-1=15$

$$z=7+\frac{1}{4}\times15=7+\frac{15}{4}=10\frac{3}{4}.$$

4. $a=\frac{1}{2}, d=-\frac{3}{2}, n-1=22$

$$z=\frac{1}{2}-\frac{3}{2}\times 22=\frac{1}{2}-33=-32\frac{1}{2}.$$

5. $a=\frac{5}{6}, d=\frac{1}{2}-\frac{5}{6}=-\frac{1}{3}, n-1=11$

$$z=\frac{5}{6}-\frac{1}{3}\times 11=\frac{5}{6}-\frac{11}{3}=-2\frac{5}{6}.$$

6. $a=-12, d=4, n-1=13$

$$z=-12+4\times 13=-12+52=40.$$

7. $a=-3, d=8, n-1=15$

$$z=-3+8\times 15=-3+120=117.$$

8. $a=\frac{n-1}{n}, d=\frac{n-2}{n}-\frac{n-1}{n}=\frac{-1}{n}, n-1=n-1$

$$z=\frac{n-1}{n}-\frac{1}{n}\times(n-1)=\frac{n-1}{n}-\frac{n-1}{n}=0.$$

9. $a=x^2+2xy+y^2, d=-2xy, n-1=n-1$

$$z=x^2+2xy+y^2-2xy(n-1)=x^2+y^2-2(n-2)xy.$$

10. $a=\frac{a-b}{a+b}, d=\frac{3a-2b}{a+b}, n-1=n-1$

$$z=\frac{a-b}{a+b}+\frac{3an-2bn-3a+2b}{a+b}=\frac{3an-2bn-2a+b}{a+b}.$$

CXXXV.

1. $2a=4, d=1, n=100$

$$s=\frac{100}{2}\{2+99\times 1\}=50\times 101=5050.$$

2. $2a=4, d=2, n=50$

$$s=25\{4+49\times 2\}=25\times 102=2550.$$

3. $2a=6,\ d=4,\ n=20$

$$s=10\{6+19\times 4\}=10\times 82=820.$$

4. $2a=\frac{1}{2},\ d=\frac{1}{4},\ n=15$

$$s=\frac{15}{2}\left\{\frac{1}{2}+\frac{1}{4}\times 14\right\}=\frac{15}{2}\times 4=30.$$

5. $2a=-18,\ d=2,\ n=12$

$$s=6\{-18+2\times 11\}=6\times 4=24.$$

6. $2a=\frac{5}{3},\ d=-\frac{1}{3},\ n=17$

$$s=\frac{17}{2}\left\{\frac{5}{3}-\frac{1}{3}\times 16\right\}=\frac{17}{2}\times\left(-\frac{11}{3}\right)=-\frac{187}{6}=-31\frac{1}{6}.$$

7. $2a=2,\ d=1,\ n=n$

$$s=\frac{n}{2}\{2+1\times(n-1)\}=\frac{n}{2}.(n+1).$$

8. $2a=2,\ d=3,\ n=n$

$$s=\frac{n}{2}\{2+3(n-1)\}=\frac{n}{2}(3n-1)=\frac{3n^2-n}{2}.$$

9. $2a=2,\ d=7,\ n=n$

$$s=\frac{n}{2}\{2+7(n-1)\}=\frac{n}{2}(7n-5)=\frac{7n^2-5n}{2}.$$

10. $2a=\frac{2(n-1)}{n},\ d=-\frac{1}{n},\ n=n$

$$s=\frac{n}{2}\left\{\frac{2n-2}{n}-\frac{n-1}{n}\right\}=\frac{n}{2}\times\frac{n-1}{n}=\frac{n-1}{2}.$$

CXXXVI.

1. $-14=100+19d;\ 19d=-114;\ d=-6.$

2. $-x=x+50d;\ 50d=-2x;\ d=-\frac{x}{25}.$

3. $5\frac{1}{2}=-\frac{1}{2}+48d$; $48d=6$; $d=\frac{1}{8}$.

4. $-21\frac{3}{4}=-\frac{3}{4}+24d$; $24d=-21$; $d=-\frac{7}{8}$.

5. $-20=-10+5d$; $5d=-10$; $d=-2$.

6. $0=150+90d$; $90d=-150$; $d=-\frac{5}{3}$.

CXXXVII.

1. (1.) $\left.\begin{array}{l} a+58d=70 \\ a+65d=84 \end{array}\right\}$; $7d=14$; $d=2$; $a=70-116=-46$.

(2.) $\left.\begin{array}{l} a+19d=93-35b \\ a+20d=98-37b \end{array}\right\}$; $d=5-2b$; $a+95-38b=93-35b$, etc.

(3.) $\left.\begin{array}{l} a+\ \ d=\frac{1}{2} \\ a+54d=5{\cdot}8 \end{array}\right\}$; $\left.\begin{array}{r} 54a+54d=27 \\ a+54d=5{\cdot}8 \end{array}\right\}$; $53a=21{\cdot}2$; $a=\frac{2}{5}$.

(4.) $\left.\begin{array}{l} a+\ \ d=4 \\ a+86d=-30 \end{array}\right\}$; $\left.\begin{array}{r} 86a+86d=344 \\ a+86d=-30 \end{array}\right\}$; $85a=374$; $a=4{\cdot}4$.

2. $\left.\begin{array}{l} (a+2d)+(a+7d)=31 \\ (a+4d)+(a+9d)=43 \end{array}\right\}$; $\left.\begin{array}{l} 2a+\ \ 9d=31 \\ 2a+13d=43 \end{array}\right\}$; $d=3$, $a=2$.

Hence sum of 10 terms $=5\{4+27\}=155$.

3. $\left.\begin{array}{l} a+(a+2d)=0 \\ (a+d)+(a+6d=40 \end{array}\right\}$; $\left.\begin{array}{l} 2a+2d=0 \\ 2a+7d=40 \end{array}\right\}$; $d=8$, $a=-8$.

Hence sum of 7 terms$=\frac{7}{2}\cdot\{-16+48\}=\frac{7}{2}\times 32=112$.

4. $\left.\begin{array}{l} a+3d=24 \\ a+4d=33 \end{array}\right\}$; $d=9$, $a=-3$.

Hence the 100th term is $-3+99\times 9=-3+891=888$.

5. $302=5+(n-1)\times 3$; $300=3n$; $n=100$.

6. $s=\frac{20}{2}\left\{32\frac{1}{6}+19\times 32\frac{1}{6}\right\}=10\times 20\times\frac{193}{6}=6433\frac{1}{3}$.

7. From the formula $s=(a+z)\frac{n}{2}$

$s=(1+103)\times 26s.=2704s.=£135.4s.$

8. (1.) 41st term $=-5+9\times 40=355$

Sum $=(-5+355)\times\frac{41}{2}=7175$.

(2.) 41st term $=4a^2-4a^2\times 40=-156a^2$

Sum $=(4a^2-156a^2)\times\frac{41}{2}=-3116a^2$.

(3.) 41st term $=1+x+(4+2x)40=161+81x$

Sum $=(1+x+161+81x)\times\frac{41}{2}=3321+1681x$.

(4.) 41st term $=-4\frac{1}{2}+3\frac{1}{10}\times 40=119\frac{1}{2}$

Sum $=\left(-4\frac{1}{2}+119\frac{1}{2}\right)\times\frac{41}{2}=2357\frac{1}{2}$.

(5.) 41st term $=\frac{1}{4}+\frac{1}{5}\times 40=8\frac{1}{4}$

Sum $=\left(\frac{1}{4}+8\frac{1}{4}\right)\times\frac{41}{2}=174\frac{1}{4}$.

9. (1.) $a=1002$; $d=-8$; nth term $=a+(n-1)d$

$2=1002-8(n-1)$; $8n=1008$; $n=126$

$s=(10002+2)\times 63=63252$.

(2.) $186=-6+8(n-1)$; $8n=200$; $n=25$

$s=(-6+186)\times\frac{25}{2}=2250$.

(3.) $-72\cdot3x=2\frac{1}{2}x-1\cdot7x(n-1)$; $1\cdot7n=76\cdot5$; $n=45$

$s=(2\cdot5x-72\cdot3x)\times\frac{45}{2}=-1570\cdot5x.$

(4.) $-24=\frac{1}{2}-(n-1)\times\frac{1}{4}$; $\frac{n}{4}=24\frac{3}{4}$; $n=99$

$s=\left(\frac{1}{2}-24\right)\times\frac{99}{2}=-\frac{4653}{4}=-1163\frac{1}{4}.$

(5.) $139(1-m)=m-1+(n-1)2(1-m)$; $139=-1+2(n-1)$; $n=71$

$s=\{m-1+139(1-m)\}\times\frac{71}{2}=4899(1-m).$

(6.) $x-2=x+254-(n-1)\times4$; $4n=2+258$; $n=65$

$s=\{x+254+x-2\}\frac{65}{2}=65x+8190.$

CXXXVIII.

1. $18=3+5d$; $d=3$; means are 6, 9, 12, 15.

2. $-2=2+6d$; $d=-\frac{2}{3}$; means are $1\frac{1}{3}, \frac{2}{3}, 0, -\frac{2}{3}, -1\frac{1}{3}$.

3. $\frac{2}{3}=3+4d$; $d=-\frac{7}{12}$; means are $2\frac{5}{12}, 1\frac{5}{6}, 1\frac{1}{4}$.

4. $\frac{1}{3}=\frac{1}{2}+5d$; $d=-\frac{1}{30}$; means are $\frac{7}{15}, \frac{13}{30}, \frac{2}{5}, \frac{11}{30}$.

CXXXIX.

1. $n=m+4d$; $d=\frac{n-m}{4}$; means are $\frac{3m+n}{4}$, etc.

2. $m-1=m+1+5d$; $d=-\frac{2}{5}$; means are $\frac{5m+3}{5}$, etc.

3. $n^2+1=n^2+5d$; $d=\frac{1}{5}$; means are $\frac{5n^2+1}{5}$, etc.

4. $x^2-y^2=x^2+y^2+4d$; $d=-\frac{y^2}{2}$; means are $\frac{2x^2+y^2}{2}$, etc.

CXL.

1. $z=1\times 2^6=64.$

2. $z=4\times 3^9=4\times 19683=78732.$

3. $z=5\times 4^8=327680.$

4. $z=8\times\left(\frac{1}{2}\right)^{14}=\frac{2^3}{2^{14}}=\frac{1}{2^{11}}=\frac{1}{2048}.$

5. $z=2\times 3^8=13122.$

6. $z=\frac{1}{64}\times 4^{10}=\frac{4^{10}}{4^3}=4^7=16384.$

7. $z=-\frac{2}{3}\times\left(-\frac{1}{2}\right)^6=\frac{-2}{3\times 2^6}=\frac{-1}{3\times 2^5}=-\frac{1}{96}.$

CXLI.

1. $s=\frac{2(2^{15}-1)}{2-1}=2(32768-1)=65534.$

2. $s=\frac{1(3^6-1)}{3-1}=\frac{729-1}{2}=364.$

3. $s=\frac{a(x^{26}-1)}{x^2-1}.$

4. $s=\frac{a\left(\frac{1}{x^9}-1\right)}{\frac{1}{x}-1}=\frac{a}{x^8}\left(\frac{x^9-1}{x-1}\right).$

5. $$s=\frac{(a^2-x^2)\left\{\frac{1}{(a+x)^7}-1\right\}}{\frac{1}{a+x}-1}=\frac{(a^2-x^2)}{(a+x)^6}\left\{\frac{1-(a+x)^7}{1-a-x}\right\}$$
$$=\frac{(a-x)\{1-(a+x)^7\}}{(a+x)^5.(1-a-x)}.$$

6. $s=\frac{2(3^n-1)}{3-1}=3^n-1.$

7. $s=\frac{7(2^n-1)}{2-1}=7(2^n-1).$

8. $s=\frac{5\{(-2)^8-1\}}{-2-1}=\frac{5(1-256)}{3}=-425.$

9. $s=\frac{-\frac{2}{3}\left\{\left(-\frac{1}{2}\right)^7-1\right\}}{-\frac{1}{2}-1}=\frac{-\frac{2}{3}\left\{-\frac{1}{128}-1\right\}}{-\frac{3}{2}}=\frac{4}{9}\times\left(-\frac{129}{128}\right)$

$=-\frac{43}{96}.$

CXLII.

1. $s=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2.$

2. $s=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}.$

3. $s=\frac{3}{1-\frac{1}{9}}=\frac{3\times 9}{8}=\frac{27}{8}.$

4. $s=\frac{\frac{2}{3}}{1-\frac{1}{2}}=\frac{4}{3}.$

5. $s=\frac{\frac{3}{4}}{1-\frac{1}{3}}=\frac{3\times 3}{4\times 2}=1\frac{1}{8}.$

6. $s=\frac{\frac{1}{2}}{1+\frac{2}{3}}=\frac{1\times 3}{2\times 5}=\frac{3}{10}.$

7. $s=\frac{8}{1-\frac{1}{12}}=\frac{8\times 12}{11}=8\frac{8}{11}.$

8. $s=\frac{1\cdot 5}{1-\frac{1}{3}}=\frac{1\cdot 5\times 3}{2}=2\frac{1}{4}.$

9. $s=\frac{64}{1-\frac{1}{4}}=\frac{64\times 4}{3}=85\frac{1}{3}.$

10. $s=\frac{2x^3}{1+\frac{1}{8x^2}}=\frac{16x^5}{8x^2+1}.$

11. $s=\frac{a}{1-\frac{b}{a}}=\frac{a^2}{a-b}.$

12. $s=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{1}{9}.$

13. $s=\dfrac{x}{1+\dfrac{y}{x}}=\dfrac{x^2}{x+y}$.

14. $s=86\left\{\dfrac{\dfrac{1}{100}}{1-\dfrac{1}{100}}\right\}=86\times\dfrac{1}{99}=\dfrac{86}{99}$.

15. $s=\dfrac{5}{10}+\dfrac{4}{100}+\dfrac{4}{1000}+\ldots=\dfrac{5}{10}+\dfrac{\dfrac{4}{100}}{1-\dfrac{1}{10}}=\dfrac{5}{10}+\dfrac{4}{90}=\dfrac{49}{90}$.

16. $s=\dfrac{8}{10}+\dfrac{36}{1000}+\dfrac{36}{100000}+\ldots=\dfrac{8}{10}+\dfrac{\dfrac{36}{1000}}{1-\dfrac{1}{100}}=\dfrac{8}{10}+\dfrac{36}{990}=\dfrac{46}{55}$.

CXLIII.

1. $243=3f^4$; $f^4=81$; $f=3$, etc.

2. $1024=1\times f^5$; $f^5=4^5$; $f=4$, etc.

3. $16=1\times f^4$; $f^4=16$; $f=2$, etc.

4. $\dfrac{243}{64}=\dfrac{1}{2}\times f^6$; $f^6=\dfrac{243}{32}$; $f=\dfrac{3}{2}$, etc.

CXLIV.

1. (1.) $s=\{16+7(12-1)\}\times\dfrac{12}{2}=93\times 6=558$.

(2.) $s=\{232-8(10-1)\}\times\dfrac{10}{2}=160\times 5=800$.

(3.) $s=\dfrac{3}{1-\dfrac{1}{6}}=\dfrac{18}{5}$.

(4.) $s=\dfrac{2}{1+\dfrac{1}{8}}=\dfrac{16}{9}.$

(5.) $s=\left\{1-\dfrac{7}{6}(13-1)\right\}\times\dfrac{13}{2}=(-13)\times\dfrac{13}{2}=-\dfrac{169}{2}.$

(6.) $s=\dfrac{\dfrac{1}{2}\cdot\left\{\dfrac{2^6}{3^6}-1\right\}}{-\dfrac{2}{3}-1}=\dfrac{3}{10}\cdot\left(\dfrac{3^6-2^6}{3^6}\right)=\dfrac{665}{2430}=\dfrac{133}{486}.$

(7.) $s=\left\{1-\dfrac{3}{2}\times 28\right\}\times\dfrac{29}{2}=(-41)\times\dfrac{29}{2}=-\dfrac{1189}{2}.$

(8.) $s=\left\{\dfrac{10}{7}+\dfrac{2}{7}\times 7\right\}\times\dfrac{8}{2}=\dfrac{24}{7}\times 4=13\dfrac{5}{7}.$

(9.) $s=\dfrac{\dfrac{1}{3}}{1-\dfrac{2}{3}}=\dfrac{1}{3}\times\dfrac{3}{1}=1.$

(10.) $s=\left\{\dfrac{6}{5}-2\times 9\right\}\times\dfrac{10}{2}=\left(-\dfrac{84}{5}\right)\times 5=-84.$

(11.) The common factor $=-\sqrt{6}\div\dfrac{\sqrt{3}}{\sqrt{5}}=-\dfrac{\sqrt{30}}{\sqrt{3}}=-\sqrt{10}$

$$s=\frac{\sqrt{\dfrac{3}{5}}\{(-\sqrt{10})^8-1\}}{-\sqrt{10}-1}=\frac{\sqrt{3}.\{10000-1\}}{\sqrt{5}\{-\sqrt{10}-1\}}=-\frac{9999\sqrt{3}}{\sqrt{5}(\sqrt{10}+1)}.$$

(12.) $s=\dfrac{-\dfrac{7}{5}\left\{-\dfrac{5^5}{2^5}-1\right\}}{-\dfrac{5}{2}-1}=-\dfrac{2}{5}\left(\dfrac{5^5+2^5}{2^5}\right)=-\dfrac{3157}{80}.$

2. Let the series be a, af, af^2, af^3, af^4.
Then $a\times af\times af^2\times af^3\times af^4=32$; $a^5f^{10}=2^5$; $af^2=2$.

3. $b=\dfrac{a+c}{2}$ and $b'=\sqrt{(ac)}$

$\therefore\ \dfrac{b}{b'}=\dfrac{a+c}{2\sqrt{(ac)}}.$

4. The arithmetic mean is $\frac{a+b}{2}$; the geometric is $\sqrt{(ab)}$.

Now since the square of every number is *positive*

$(\sqrt{a}-\sqrt{b})^2$ is greater than 0

$a-2\sqrt{(ab)}+b$ is greater than 0.

$\therefore \frac{a+b}{2}$ is greater than $\sqrt{(ab)}$.

5. $a+(a+d)+(a+2d)=12$; $3a+3d=12$; $a+d=4$.
Also, $a+5d=12$. Hence $4d=8$; $d=2$; $a=2$.
Then sum of 6 terms $=\{4+10\}\times 3=42$.

6. Let f be the common factor: $af=b$ and $af^2=c$
$\therefore ac=a^2f^2=b^2$.

7. $2n\times\frac{1}{2n}=x^2$; $1=x^2$; $x=\pm 1$.

8. $2n+\frac{1}{2n}=2y$; $y=n+\frac{1}{4n}$.

9. The sum of the geometric progression is $\frac{3^4-1}{3-1}$ or $\frac{80}{2}$ or 40.

The sum of the arithmetic progression is $\{8+4(n-1)\}\frac{n}{2}$.

Hence $2n+2n^2=40$; $n^2+n=20$; whence $n=4$.

10. The first term is 1, the constant difference 1.

Hence $153=\{2+(7+n-1)\}\frac{7+n}{2}$; $306=(8+n)(7+n)$;

$n^2+15n=250$; $n=10$.

11. Let n be the number of terms.

Then $\{2+2(n-1)\}\frac{n}{2}=2n\times\frac{n}{2}=n^2$.

12. Let the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$.
Then $5a+10d=95$; and $\therefore a+2d=19$.

13. $22 = \left\{6\frac{2}{3} + \frac{13}{9}(n-1)\right\}\frac{n}{2}$; $44 = n\left(\frac{47+13n}{9}\right)$;
$396 = 47n + 13n^2$; whence $n = 4$.

14. Let $100x + 10y + z$ be the number.
Then $x + z = 2y$ (1)
and $\frac{100x + 10y + z}{y + z} = 107$; or $100x - 97y - 106z = 0$ (2)
and $100x + 10y + z - 396 = 100z + 10y + x$. (3)
From (3) $99x - 396 = 99z$; or $x - z = 4$.
From this and (1) $x = y + 2$ and $z = y - 2$.
Hence from (2) $100y + 200 - 97y - 106y + 212 = 0$.
Thus $y = 4$, $x = 6$, $z = 2$.

15. In any geometrical series a, af, etc.
the $(p+q)$th term is af^{p+q-1}
the $(p-q)$th term is af^{p-q-1}
the pth term is af^{p-1}.
Hence $mn = af^{p+q-1} \times af^{p-q-1} = a^2f^{2p-2} = (af^{p-1})^2$
$\therefore af^{p-1} = \sqrt{(mn)}$.

16. Let x and y be the numbers. Then

$$\left.\begin{array}{l} x - y = 48 \\ \frac{x+y}{2} = \sqrt{(xy)} + 18 \end{array}\right\}; \quad \left.\begin{array}{l} x - y = 48 \\ x + y - 2\sqrt{xy} = 36 \end{array}\right\}; \quad \left.\begin{array}{r} x - y = 48 \\ \sqrt{x} - \sqrt{y} = 6 \end{array}\right\};$$

dividing, $\sqrt{x} + \sqrt{y} = 8$; hence $2\sqrt{x} = 14$; $\sqrt{x} = 7$; $x = 49$; $y = 1$

17. $11 = 1 + 4d$; $4d = 10$; $d = \frac{5}{2}$, etc.

18. $2{\cdot}748 = \{{\cdot}068 + (n-1) \times {\cdot}0004\} \times \frac{n}{2}$; $5{\cdot}496 = {\cdot}0676n + {\cdot}0004n^2$;
$54960 = 4n^2 + 676n$; $n^2 + 169n = 13740$; $n = 60$.

19. $-1 = 1 + 10d$; $10d = -2$; $d = -\frac{1}{5}$, etc.

20. The $(n+1)$th term is 2^n.

The sum of n terms is $\frac{2^n-1}{2-1}=2^n-1$.

21. Sum of the *first* p terms $=\frac{a(r^p-1)}{r-1}$.

Sum of the *second* p terms $=\frac{ar^p(r^p-1)}{r-1}$.

Sum of the *third* p terms $=\frac{ar^{2p}(r^p-1)}{r-1}$, and so on, the sums forming a geometric series whose common factor is r^p.

22. $$\left.\begin{array}{l}(a-2d)+(a-d)+a+(a+d)+(a+2d)=55\\(a-2d)^2+(a-d)^2+a^2+(a+d)^2+(a+2d)^2=765\end{array}\right\};$$

$$\left.\begin{array}{l}5a=55\\5a^2+10d^2=765\end{array}\right\};\ a=11;\ d=4,\text{ etc.}$$

23. Let the series be a, af, af^2, af^3, af^4.

$$\left.\begin{array}{l}af^4-a:af^3-af=10:3\\af+af^3=2a^2f\end{array}\right\};\ \left.\begin{array}{l}3f^4-3=10f^3-10f\\1+f^2=2a\end{array}\right\};$$

$$\left.\begin{array}{l}3(f^4-1)=10f(f^2-1)\\1+f^2=2a\end{array}\right\};\ \left.\begin{array}{ll}3(f^2+1)=10f & (1)\\1+f^2=2a & (2)\end{array}\right\};$$

from (1) we find $f=3$; and then from (2) $a=5$, etc.

24. This is explained in Art. 472.

25. Let $100x+10y+z$ be the number. Then

$$\left.\begin{array}{l}y^2=xz\\x+y+z=13\\100x+10y+z+792=100z+10y+x\end{array}\right\}$$

From the third equation we get $x-z=-8$.

Hence $x^2-2xz+z^2=64$.

But $\quad 4xz \quad =4y^2$

$\therefore (x+z)^2=64+4y^2$.

Also $x+z=13-y$

$\therefore (13-y)^2=64+4y^2$.

Hence we get $y=3$; and then $x=1$ and $z=9$.

26. Let x be the increase per cent. Then the population of any year is $\frac{100+x}{100}$ times the population of the preceding year.

Hence $10000 \times \left(\frac{100+x}{100}\right)^4 = 14641$.

Take the fourth root of each side; then

$10 \times \frac{100+x}{100} = 11$; $100+x=110$; $x=10$.

CXLV.

1. First insert two arithmetic means between $\frac{1}{6}$ and $\frac{1}{24}$.

$\frac{1}{24}=\frac{1}{6}+3d$; $3d=-\frac{3}{24}$; $d=-\frac{1}{24}$.

Hence the arithmetic means are $\frac{1}{8}$ and $\frac{1}{12}$

∴ the harmonic means are 8 and 12.

2. $\frac{1}{3}=\frac{1}{2}+5d$; $5d=-\frac{1}{6}$; $d=-\frac{1}{30}$.

Hence the arithmetic means are $\frac{7}{15}$, $\frac{13}{30}$, $\frac{2}{5}$, $\frac{11}{30}$

∴ the harmonic means are $\frac{15}{7}$, $\frac{30}{13}$, $\frac{5}{2}$, $\frac{30}{11}$.

3. $\frac{2}{3}=3+4d$; $4d=-\frac{7}{3}$; $d=-\frac{7}{12}$.

Hence the arithmetic means are $\frac{29}{12}$, $\frac{11}{6}$, $\frac{5}{4}$

∴ the harmonic means are $\frac{12}{29}$, $\frac{6}{11}$, $\frac{4}{5}$.

4. $18=3+5d$; $5d=15$; $d=3$.

Hence the arithmetic means are 6, 9, 12, 15

∴ the harmonic means are $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, $\frac{1}{15}$.

5. $2^{-1}=\frac{1}{2}$; and we have to insert five arithmetic means between -1 and 2.

$2=-1+6d$; $6d=3$; $d=\frac{1}{2}$.

Hence the arithmetic means are $-\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$

$\therefore$ the harmonic means are $-2, \infty, 2, 1, \frac{2}{3}$.

6. $-2=2+6d$; $6d=-4$; $d=-\frac{2}{3}$.

Hence the arithmetic means are $\frac{4}{3}, \frac{2}{3}, 0, -\frac{2}{3}, -\frac{4}{3}$

$\therefore$ the harmonic means are $\frac{3}{4}, \frac{3}{2}, \infty, -\frac{3}{2}, -\frac{3}{4}$.

7. $\frac{23}{6}=\frac{1}{3}+7d$; $7d=\frac{21}{6}$; $d=\frac{1}{2}$.

Hence the arithmetic means are $\frac{5}{6}, \frac{4}{3}, \frac{11}{6}, \frac{7}{3}, \frac{17}{6}, \frac{10}{3}$

$\therefore$ the harmonic means are $\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

8. $\frac{1}{3y}=\frac{1}{2x}+(n+1)d$; $d=\frac{2x-3y}{6xy(n+1)}$.

First arithmetic mean is $\frac{1}{2x}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+3y+2x-3y}{6xy(n+1)}$

$=\frac{3ny+2x}{6xy(n+1)}$.

Second arithmetic mean is $\frac{3ny+2x}{6xy(n+1)}+\frac{2x-3y}{6xy(n+1)}=\frac{3ny+4x-3y}{6xy(n+1)}$.

. . . .

nth arithmetic mean is $\frac{1}{3y}-\frac{2x-3y}{6xy(n+1)}=\frac{2nx+3y}{6xy(n+1)}$.

Hence the harmonic means are $\frac{6xy(n+1)}{3ny+2x}, \frac{6xy(n+1)}{3ny+4x-3y}, \dots$

$\frac{6xy(n+1)}{2nx+3y}$.

9. Let $\frac{1}{x}$ and $\frac{1}{y}$ be the second and third terms.

Then 2, x, y are in arithmetic progression.

$$\left.\begin{array}{l}\text{Hence } 2+y=2x \\ \text{and } \frac{1}{2}+\frac{1}{x}+\frac{1}{y}=\frac{11}{12}\end{array}\right\}; \quad \frac{1}{2}+\frac{1}{x}+\frac{1}{2x-2}=\frac{11}{12}$$

$$\frac{2x-2+x}{2x^2-2x}=\frac{5}{12}; \quad 36x-24=10x^2-10x; \quad 10x^2-46x=-24:$$

hence we find $x=4$ or $\frac{3}{5}$; and $\therefore y=6$ or $-\frac{4}{5}$.

The arithmetic progressions will therefore be

$$-4, \ -2, \ 0, \ 2, \ 4, \ 6; \text{ and } \frac{31}{5}, \ \frac{24}{5}, \ \frac{17}{5}, \ 2, \ \frac{3}{5}, -\frac{4}{5}.$$

$\therefore$ the harmonic progressions are

$$-\frac{1}{4}, \ -\frac{1}{2}, \ \infty, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{6} \text{ and } \frac{5}{31}, \ \frac{5}{24}, \ \frac{5}{17}, \ \frac{1}{2}, \ \frac{5}{3}, \ -\frac{5}{4}.$$

10. Let x and y be the numbers. Then

$$\frac{x+y}{2}=\sqrt{(xy)}+13 \ \ (1); \text{ and } \sqrt{(xy)}=\frac{2xy}{x+y}+12$$

$$\frac{2xy}{x+y}=\sqrt{(xy)}-12 \ \ (2).$$

Multiply (1) by (2); $xy=xy+\sqrt{(xy)}-156$.

Hence $\sqrt{(xy)}=156$; from this and (1) we find $x=104$, $y=234$.

11. $2b=a+c$ and $\frac{1}{b}+\frac{1}{d}=\frac{2}{c}$

$$\therefore 2b\left(\frac{1}{b}+\frac{1}{d}\right)=\frac{2}{c}(a+c); \quad 2+\frac{2b}{d}=\frac{2a}{c}+2;$$

$\therefore bc=ad$, and $\therefore a:b=c:d$.

12. $$x=\frac{2mn}{m+n}; \quad \frac{1}{x-m}+\frac{1}{x-n}=\frac{1}{\frac{2mn}{m+n}-m}+\frac{1}{\frac{2mn}{m+n}-n}$$

$$=\frac{m+n}{mn-m^2}+\frac{m+n}{mn-n^2}=\frac{m+n}{n-m}\left\{\frac{1}{m}-\frac{1}{n}\right\}=\frac{m+n}{mn}=\frac{1}{m}+\frac{1}{n}.$$

13. $$\left.\begin{array}{l} x+y+z=11\\ x^2+y^2+z^2=49\\ y=\dfrac{2xz}{x+z}\end{array}\right\};\quad \left.\begin{array}{l}(x+z)^2=121-22y+y^2\\ x^2+z^2=49-y^2\\ 2xz=72-22y+2y^2\end{array}\right\}$$

Now $(x+z)y=2xz$.

$\therefore\ (11-y)y=72-22y+2y^2$; whence $y=3$.

Then $x+z=8$, and $xz=12$; whence $x=2$, $z=6$.

14. $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are the pth, qth, and rth terms of an arithmetic progression, suppose $a, a+d. \ . \ . \ .$

Then $$\left.\begin{array}{l} \dfrac{1}{x}=a+(p-1)d\\ \dfrac{1}{y}=a+(q-1)d\\ \dfrac{1}{z}=a+(r-1)d\end{array}\right\};\quad \begin{array}{l}\dfrac{1}{x}-\dfrac{1}{z}=(p-r)d\\ \dfrac{1}{z}-\dfrac{1}{y}=(r-q)d\\ \dfrac{1}{y}-\dfrac{1}{x}=(q-p)d\end{array}$$

Hence $$\left.\begin{array}{l} z-x=(p-r)xzd\\ y-z=(r-q)yzd\\ x-y=(q-p)xyd\end{array}\right\};$$

adding $0=(p-r)xzd+(r-q)yzd+(q-p)xyd$

$\therefore\quad 0=(p-r)xz+(r-q)yz+(q-p)xy.$

15. $\dfrac{2ab}{a+b}, \dfrac{2bc}{b+c}, \dfrac{2ca}{c+a}$ are in A. P.

$$\therefore\ \frac{bc}{b+c}-\frac{ab}{a+b}=\frac{ca}{c+a}-\frac{bc}{b+c}$$

$$\therefore\ \frac{b^2c-ab^2}{(b+c)(a+b)}=\frac{c^2a-bc^2}{(b+c)(c+a)},\ \text{or,}\ \frac{b^2(c-a)}{a+b}=\frac{c^2(a-b)}{c+a}$$

$\therefore\ b^2(c^2-a^2)=c^2(a^2-b^2)$, or, $2b^2c^2=a^2b^2+a^2c^2$

$\therefore\ \dfrac{2}{a^2}=\dfrac{1}{c^2}+\dfrac{1}{b^2}$; $\because\ b^2, a^2, c^2$ are in H. P.

Again, $\dfrac{a+b}{2ab}, \dfrac{b+c}{2bc}, \dfrac{c+a}{2ca}$ are in A. P.

$$\therefore \frac{b+c}{bc}-\frac{a+b}{ab}=\frac{c+a}{ac}-\frac{b+c}{bc}$$

$$\therefore \frac{1}{c}+\frac{1}{b}-\frac{1}{b}-\frac{1}{a}=\frac{1}{a}+\frac{1}{c}-\frac{1}{c}-\frac{1}{b}$$

$$\therefore \frac{1}{c}+\frac{1}{b}=\frac{2}{a}\,;\ \therefore b,\ a,\ c \text{ are in H. P.}$$

16. (1.) When c is the arithmetic mean between a and b.

$$\frac{\frac{a+b}{2}+2a}{\frac{a+b}{2}-b}+\frac{\frac{a+b}{2}+2b}{\frac{a+b}{2}-a}=\frac{5a+b}{a-b}+\frac{a+5b}{b-a}$$

$$=\frac{5a+b-a-5b}{a-b}=\frac{4a-4b}{a-b}=4.$$

(2.) When c is the geometric mean between a and b.

$$\frac{\sqrt{(ab)}+2a}{\sqrt{(ab)}-b}+\frac{\sqrt{(ab)}+2b}{\sqrt{(ab)}-a}$$

$$=\frac{\sqrt{a}(\sqrt{b}+2\sqrt{a})}{\sqrt{b}(\sqrt{a}-\sqrt{b})}-\frac{\sqrt{b}(\sqrt{a}+2\sqrt{b})}{\sqrt{a}(\sqrt{a}-\sqrt{b})}=\frac{a\sqrt{b}+2a\sqrt{a}-b\sqrt{a}-2b\sqrt{b}}{\sqrt{(ab)}.(\sqrt{a}-\sqrt{b})}$$

$$=\frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}}\,;\text{ and since } a+b \text{ is greater than } 2\sqrt{(ab)},$$

$$\therefore \frac{2a+2b+3\sqrt{(ab)}}{\sqrt{(ab)}} \text{ is greater than } \frac{4\sqrt{(ab)}+3\sqrt{(ab)}}{\sqrt{(ab)}}\text{, or, than 7.}$$

(3.) When c is the harmonic mean between a and b.

$$\frac{\frac{2ab}{a+b}+2a}{\frac{2ab}{a+b}-b}+\frac{\frac{2ab}{a+b}+2b}{\frac{2ab}{a+b}-a}=\frac{4ab+2a^2}{ab-b^2}+\frac{4ab+2b}{ab-a^2}$$

$$=\frac{4a^2b+2a^3-4ab^2-2b^3}{ab(a-b)}=\frac{2a^2+6ab+2b^2}{ab}.$$

Now a^2+b^2 is greater than $2ab$

$$\therefore \frac{2a^2+2b^2+6ab}{ab} \text{ is greater than } \frac{4ab+6ab}{ab}\text{, or, than 10.}$$

CXLVI.

1. $n.(n-1)=12\times 11=132$.

2. $n.(n-1).(n-2)=16\times 15\times 14=3360$.

3. $n.(n-1).(n-2).(n-3)=20\times 19\times 18\times 17=116280$.

4. $n.(n-1).(n-2)(n-3)(n-4)=8\times 7\times 6\times 5\times 4=6720$.

5. Number of letters is 11, and three are twice repeated,

$\therefore$ number of permutations $=\dfrac{\underline{|11}}{\underline{|2}.\underline{|2}.\underline{|2}}=4989600$.

6. $n.(n-1)\ .\ .\ .\ (n-7)=8\times 7\times 6\times 5\times 4\times 3\times 2\times 1=40320$.

7. $n.(n-1)\ .\ .\ .\ (n-9)=\underline{|10}=3628800$.

8. (1.) The number of signals in which we have 3 flags of different colours is $5.4.3=60$.

(2.) When 2 flags in a signal are of the same colour, suppose red, we can put 4 different colours with these, and each of the resulting signals may be arranged in 3 different ways, that is, with both the red first, or both last, or one first and one last. Thus we shall have 12 different signals with two reds. Similarly we shall have 12 different signals with two of each of the other 4 colours.

$\therefore$ we shall have 5×12 or 60 in all.

(3.) When we have 3 of the same colour, we shall have 5 different signals in all.

$\therefore$ total number $=60+60+5=125$.

9. Number of permutations $=\dfrac{\underline{|7}}{\underline{|2}}=\dfrac{5040}{2}=2520$.

10. $n:n(n-1)(n-2)=1:20$;

$20n=n(n-1)(n-2)$; $n^2-3n+2=20$; $n=6$.

11. $m.(m-1)(m-2):(m+2)(m+1)m=1:5$;
$5m.(m-1)(m-2)=(m+2)(m+1)m$;
$5m^2-15m+10=m^2+3m+2$; $m=4$.

12. Number of letters is 7, and therefore the number of permutations in which *cd* stand first$=\underline{|5}=120$.

13. Number of letters is 9, one of which is repeated twice, one three times, and one four times.

$$\therefore \text{ Number of permutations }=\frac{\underline{|9}}{\underline{|2}.\underline{|3}.\underline{|4}}=9\times4\times7\times5=1260.$$

14. In *Conceit* we have 7 letters, one of which occurs twice,

$$\therefore \text{ number of permutations }=\frac{\underline{|7}}{\underline{|2}}=2520.$$

In *Talavera*, 8 letters, one of which is repeated three times,

$$\therefore \text{ number of permutations }=\frac{\underline{|8}}{\underline{|3}}=6720.$$

In *Calcutta*, 8 letters, 3 of them repeated twice,

$$\therefore \text{ number of permutations }=\frac{\underline{|8}}{2\times2\times2}=5040.$$

In *Proposition*, 11 letters, 2 repeated twice, 1 thrice,

$$\therefore \text{ number of permutations }=\frac{\underline{|11}}{\underline{|2}.\ \underline{|2}.\ \underline{|3}}=1663200.$$

In *Mississippi*, 11 letters, 2 repeated 4 times, 1 twice,

$$\therefore \text{ number of permutations }=\frac{\underline{|11}}{\underline{|4}.\ \underline{|4}.\ \underline{|2}}=34650.$$

CXLVII.

1. ${}_{100}C_4=\dfrac{100\times99\times98\times97}{1\times2\times3\times4}=3921225.$

2. ${}_6C_5=\dfrac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}=6.$

3. Leave out a, and find how many combinations can be formed out of the remaining 9 letters, 4 at a time

${}_9C_4=\dfrac{9\times8\times7\times6\times5}{1\times2\times3\times4\times5}=126$, with each of which a will combine to form a word of 5 letters.

4. Out of the 19 consonants we get $\dfrac{19\times18\times17}{1\times2\times3}$ combinations of 3,

then $\dfrac{19\times18\times17}{1\times2\times3}\times5$ = number of ways in which we can get 3 consonants and 1 vowel, and each combination of 4 letters admits of $|\underline{4}$ permutations.

$\therefore$ number of words $=\dfrac{19.18.17}{1.2.3}\times5\times|\underline{4}=116280.$

5. $\dfrac{n.(n-1)(n-2)(n-3)}{1.2.3.4}:\dfrac{n.(n-1)}{1.2}=15:2$; $(n-2)(n-3)=90$;

$n=12.$

6. $\dfrac{n.(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}=\dfrac{18}{5}\times\dfrac{n.(n-1)(n-2)}{1.2.3}$;

$(n-3)(n-4)=18\times4$; $n=12.$

7. Number of words $=\dfrac{17.16.15}{1.2.3}\times\dfrac{5.4}{1.2}\times|\underline{5}=816000.$

8. Number $=\dfrac{12.11.10.9.8.7}{1.2.3.4.5.6}\times\dfrac{5.4.3}{1.2.3}\times|\underline{9}=3353011200.$

9. $n(n-1)(n-2)=\dfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}\times6$; $6(n-3)=24$;

$n-3=4$; $n=7.$

10. Taking the coins singly, by twos, by threes, and so on,

number of combinations $=6+\dfrac{6.5}{1.2}+\dfrac{6.5.4}{1.2.3}+\dfrac{6.5.4.3}{1.2.3.4}+\dfrac{6.5.4.3.2}{1.2.3.4.5}+1$

$=6+15+20+15+6+1=63.$

11. $\dfrac{n.(n-1)(n-2)}{1.2.3}=425\times n$; $(n-1)(n-2)=2550$; hence $n=52.$

12. Number $=\frac{12.11.10}{1.2.3}\times\frac{16.15.14}{1.2.3}=123200.$

13. (1.) The number is $\frac{36.35.34.33.32}{1.2.3.4.5}=376992.$

(2.) He will go out with as many different parties as can be formed by taking 4 out of 35.

$\therefore$ he goes with $\frac{35.34.33.32}{1.2.3.4}=52360.$

CXLVIII.

1. $a^4+4a^3x+\frac{4.3}{1.2}a^2x^2+\frac{4.3.2}{1.2.3}ax^3+x^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$

2. $b^6+6b^5c+\frac{6.5}{1.2}b^4c^2+\frac{6.5.4}{1.2.3}b^3c^3+\frac{6.5.4.3}{1.2.3.4}b^2c^4+\frac{6.5.4.3.2}{1.2.3.4.5}bc^5+c^6$
$=b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6.$

3. $a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\frac{7.6.5.4}{1.2.3.4}a^3b^4+\frac{7.6.5.4.3}{1.2.3.4.5}a^2b^5$
$+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}ab^6+b^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$
$+21a^2b^5+7ab^6+b^7.$

4. $x^8+8x^7y+\frac{8.7}{1.2}x^6y^2+\frac{8.7.6.}{1.2.3.}x^5y^3+\frac{8.7.6.5}{1.2.3.4}x^4y^4+\frac{8.7.6.5.4}{1.2.3.4.5}x^3y^5$
$+\frac{8.7.6.5.4.3}{1.2.3.4.5.6}x^2y^6+\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}xy^7+y^8$
$=x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8.$

5. $5^4+4.5^3.4a+\frac{4.3}{1.2}.5^2.4^2a^2+\frac{4.3.2}{1.2.3}.5.4^3a^3+(4a)^4$
$=625+2000a+2400a^2+1280a^3+256a^4.$

6. $(a^2)^5+5(a^2)^4.bc+\frac{5.4}{1.2}(a^2)^3.(bc)^2+\frac{5.4.3}{1.2.3}(a^2)^2.(bc)^3+\frac{5.4.3.2}{1.2.3.4}a^2(bc)^4$
$+(bc)^5=a^{10}+5a^8bc+10a^6b^2c^2+10a^4b^3c^3+5a^2b^4c^4+b^5c^5.$

CXLIX.

1. $a^6-6a^5x+\frac{6.5}{1.2}.a^4x^2-\frac{6.5.4}{1.2.3}a^3x^3+\frac{6.5.4.3}{1.2.3.4}a^2x^4-\frac{6.5.4.3.2}{1.2.3.4.5}ax^5+x^6$

$=a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6.$

2. $b^7-7b^6c+\frac{7.6}{1.2}b^5c^2-\frac{7.6.5}{1.2.3}b^4c^3+\frac{7.6.5.4}{1.2.3.4}.b^3c^4-\frac{7.6.5.4.3}{1.2.3.4.5}b^2c^5$

$+\frac{7.6.5.4.3.2}{1.2.3.4.5.6}\cdot bc^6-c^7$

$=b^7-7b^6c+21b^5c^2-35b^4c^3+35b^3c^4-21b^2c^5+7bc^6-c^7.$

3. $(2x-3y)^5=\left\{2x\left(1-\frac{3y}{2x}\right)\right\}^5$

$=32x^5\left\{1-5.\frac{3y}{2x}+\frac{5.4}{1.2}\cdot\left(\frac{3y}{2x}\right)^2-\frac{5.4.3}{1.2.3}\cdot\left(\frac{3y}{2x}\right)^3\right.$

$\left.+\frac{5.4.3.2}{1.2.3.4}\cdot\left(\frac{3y}{2x}\right)^4-\left(\frac{3y}{2x}\right)^5\right\}$

$=32x^5\left\{1-\frac{15}{2}\cdot\frac{y}{x}+\frac{5.2.9}{4}\cdot\frac{y^2}{x^2}-\frac{5.2.27}{8}\cdot\frac{y^3}{x^3}\right.$

$\left.+\frac{5.81}{16}\cdot\frac{y^4}{x^4}-\frac{243}{32}\cdot\frac{y^5}{x^5}\right\}$

$=32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5.$

4. $1-5(2x)+\frac{5.4}{1.2}(2x)^2-\frac{5.4.3}{1.2.3}(2x)^3+\frac{5.4.3.2}{1.2.3.4}(2x)^4-(2x)^5$

$=1-10x+40x^2-80x^3+80x^4-32x^5.$

5. $1-10x+\frac{10.9}{1.2}x^2-\frac{10.9.8}{1.2.3}x^3+\frac{10.9.8.7}{1.2.3.4}.x^4-\frac{10.9.8.7.6}{1.2.3.4.5}x^5$

$+\frac{10.9.8.7}{1.2.3.4}x^6-\frac{10.9.8}{1.2.3}x^7+\frac{10.9}{1.2}\cdot x^8-10x^9+x^{10}$

$=1-10x+45x^2-120x^3+210x^4-252x^5+210x^6-120x^7$

$+45x^8-10x^9+x^{10}.$

6. $a^{24} - 8a^{21}b^2 + \frac{8.7}{1.2}a^{18}b^4 - \frac{8.7.6}{1.2.3}a^{15}b^6 + \frac{8.7.6.5}{1.2.3.4}a^{12}b^8 - \frac{8.7.6}{1.2.3}a^9b^{10}$
$+\frac{8.7}{1.2}a^6b^{12} - 8a^3b^{14} + b^{16}$
$= a^{24} - 8a^{21}b^2 + 28a^{18}b^4 - 56a^{15}b^6 + 70a^{12}b^8 - 56a^9b^{10}$
$+ 28a^6b^{12} - 8a^3b^{14} + b^{16}$.

CL.

1. $\{(a+2b)-c\}^3 = (a+2b)^3 - 3(a+2b)^2.c + 3(a+2b).c^2 - c^3$
$= a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3$.

2. $\{(1-2x)+3x^2\}^3 = (1-2x)^3 + 3(1-2x)^2.\ 3x^2 + 3(1-2x).9x^4 + 27x^6$
$= 1 - 6x + 12x^2 - 8x^3 + 9x^2 - 36x^3 + 36x^4 + 27x^4 - 54x^5 + 27x^6$
$= 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

3. $(x^3 - x^2 + x)^3 = x^3(x^2 - x + 1)^3 = x^3\{(1-x) + x^2\}^3$
$= x^3\{(1-x)^3 + 3(1-x)^2.x^2 + 3(1-x).x^4 + x^6\}$
$= x^3\{1 - 3x + 3x^2 - x^3 + 3x^2 - 6x^3 + 3x^4 + 3x^4 - 3x^5 + x^6\}$
$= x^9 - 3x^8 + 6x^7 - 7x^6 + 6x^5 - 3x^4 + x^3$.

4. $\{(1+2x^{\frac{1}{6}}) + 3x^{\frac{1}{3}}\}^3 = (1+2x^{\frac{1}{6}})^3 + 3(1+2x^{\frac{1}{6}})^2.3x^{\frac{1}{3}} + 3(1+2x^{\frac{1}{6}}).9x^{\frac{2}{3}} + 27x$
$= 1 + 6x^{\frac{1}{6}} + 12x^{\frac{1}{3}} + 8x^{\frac{1}{2}} + 9x^{\frac{1}{3}} + 36x^{\frac{1}{2}} + 36x^{\frac{2}{3}} + 27x^{\frac{2}{3}} + 54x^{\frac{5}{6}} + 27x$
$= 27x + 54x^{\frac{5}{6}} + 63x^{\frac{2}{3}} + 44x^{\frac{1}{2}} + 21x^{\frac{1}{3}} + 6x^{\frac{1}{6}} + 1$.

5. $\left\{(x+1) - \frac{1}{x}\right\}^3 = (x+1)^3 - 3(x+1)^2.\frac{1}{x} + 3(x+1).\frac{1}{x^2} - \frac{1}{x^3}$
$= x^3 + 3x^2 + 3x + 1 - 3x - 6 - \frac{3}{x} + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$
$= x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$.

6. $\{(a^{\frac{1}{4}} + b^{\frac{1}{4}}) - c^{\frac{1}{4}}\}^3 = (a^{\frac{1}{4}} + b^{\frac{1}{4}})^3 - 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})^2.c^{\frac{1}{4}} + 3(a^{\frac{1}{4}} + b^{\frac{1}{4}})c^{\frac{1}{2}} - c^{\frac{3}{4}}$
$= a^{\frac{3}{4}} + 3a^{\frac{1}{2}}b^{\frac{1}{4}} + 3a^{\frac{1}{4}}b^{\frac{1}{2}} + b^{\frac{3}{4}} - 3a^{\frac{1}{2}}c^{\frac{1}{4}} - 6a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}} - 3b^{\frac{1}{2}}c^{\frac{1}{4}} + 3a^{\frac{1}{4}}c^{\frac{1}{2}}$
$+ 3b^{\frac{1}{4}}c^{\frac{1}{2}} - c^{\frac{3}{4}}$.

CLI.

1. The 8th term of $(1+x)^{11}$ is $\frac{11.10.9.8.7.6.5}{1.2.3.4.5.6.7}.x^7 = 330x^7$.

2. The 5th term of $(a^2-b^2)^{12}$ is $\frac{12.11.10.9}{1.2.3.4}.(a^2)^{12-5+1}.(b^2)^4 = 495a^{16}.b^8$.

3. The 4th term of $(a-b)^{100}$ is $-\frac{100.99.98}{1.2.3}.a^{100-4+1}.b^3 = -161700a^{97}b^3$.

4. The 9th term of $(2ab-cd)^{14}$ is $\frac{14.13.12.11.10.9.8.7}{1.2.3.4.5.6.7.8}.(2ab)^{14-9+1}.(cd)^8$

$= 192192a^6b^6c^8d^8$.

5. The 9th term of $(a-b)^{16}$ is $\frac{16.15.14.13.12.11.10.9}{1.2.3.4.5.6.7.8}.a^{16-9+1}.b^8$

$= 12870a^8b^8$.

6. The 5th term of $(a^{\frac{1}{3}}+b^{\frac{1}{3}})^8$ is $\frac{8.7.6.5}{1.2.3.4}.(a^{\frac{1}{3}})^{8-5+1}.(b^{\frac{1}{3}})^4 = 70a^{\frac{4}{3}}b^{\frac{4}{3}}$.

7. The 10th term of $(a-b)^{19}$ is $-\frac{19.18.17.16.15.14.13.12.11}{1.2.3.4.5.6.7.8.9}.a^{10}.b^9$

$= -92378a^{10}b^9$.

The 11th term of $(a-b)^{19}$ is therefore $92378a^9b^{10}$.

8. The 7th term of $(a+x)^{13}$ is $\frac{13.12.11.10.9.8}{1.2.3.4.5.6}.a^7.x^6 = 1716a^7x^6$.

The 8th term of $(a+x)^{13}$ is therefore $1716a^6x^7$.

9. The coefficient of the rth term of $(a+x)^n$ is $\frac{n.(n-1)\ldots(n-r+2)}{1.2\ .\ .\ .\ (r-1)}$,

and the middle term of $(a+x)^{4n}$ is the $(2n+1)$th.

$$\therefore \text{ Coefficient } = \frac{4n.(4n-1)\ .\ .\ .\ (4n-2n-1+2)}{1.2\ .\ .\ .\ (2n+1-1)}$$

$$= \frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-1)\ .\ .\ .\ (2n+1)}{1.2\ .\ .\ .\ 2n}\cdot\frac{2n.(2n-1)\ .\ .\ .\ 1}{1.2\ .\ .\ .\ 2n}$$

$$=\frac{4n.(4n-2)\ .\ .\ .\ 6.4.2}{2n.(2n-1)\ .\ .\ .\ 3.2.1}\cdot\frac{(4n-1)(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2.2\ .\ .\ .\ \text{to } 2n \text{ factors}.\ \frac{(4n-1).(4n-3)\ .\ .\ .\ 5.3.1}{1.2.3\ .\ .\ .\ 2n}$$

$$=2^{2n}.\ \frac{1.3.5\ .\ .\ .\ (4n-1)}{1.2.3\ .\ .\ .\ 2n}.$$

10. The middle term is the $(2n+2)$th.

$$\therefore\ \text{Coefficient }=\frac{(4n+2)(4n+1)\ .\ .\ .\ (4n+2-2n-2+2)}{1.2\ .\ .\ .\ (2n+2-1)}$$

$$=\frac{(4n+2).(4n+1)\ .\ .\ .\ (2n+2)}{1.2\ .\ .\ .\ (2n+1)}$$

$$=\frac{(4n+2)(4n)(4n-2)...(2n+4)(2n+2)}{(2n+1)(2n)(2n-1)...(n+1)}.\frac{(4n+1)(4n-1)...(2n+3)}{n.(n-1)...2.1}$$

$$=2.2\ .\ .\ .\ \text{to } (n+1) \text{ factors}.\ \frac{(4n+1)(4n-1)\ .\ .\ .\ (2n+3)}{n.(n-1)\ .\ .\ .\ 2.1}$$

$$=2^{n+1}.\ \frac{(2n+3)(2n+5)\ .\ .\ .\ (4n-1)(4n+1)}{1.2\ .\ .\ .\ n}.$$

CLII.

1. $$1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}x^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^4$$

$$1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4.$$

2. $$1+\frac{2}{3}a+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1.2}a^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{1.2.3}a^3$$

$$=1+\frac{2a}{3}-\frac{a^2}{9}+\frac{4a^3}{81}.$$

3. $$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{x}{a}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\cdot\frac{x^3}{a^3}\right.$$

$$\left.+\frac{\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{1.2.3.4}\cdot\frac{x^4}{a^4}\right\}$$

$$=a^{\frac{1}{3}}+\frac{x}{3a^{\frac{2}{3}}}-\frac{x^2}{9a^{\frac{5}{3}}}+\frac{5x^3}{81a^{\frac{8}{3}}}-\frac{10x^4}{243a^{\frac{11}{3}}}.$$

4. $$1+x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}\cdot(2x)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}(2x)^3$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}(2x)^4$$

$$=1+x-\frac{x^2}{2}+\frac{x^3}{2}-\frac{5x^4}{8}.$$

5. $$a^{\frac{3}{4}}\left\{1+\frac{x}{a}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}\cdot\left(\frac{4x}{3a}\right)^2+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot\left(\frac{4x}{3a}\right)^3\right\}$$

$$=a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{1}{6}x^{-\frac{5}{4}}x^2+\frac{5}{54}a^{-\frac{9}{4}}.x^3.$$

6. $$a^{\frac{1}{5}}\left\{1+\frac{4}{5}\cdot\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)}{1.2}\cdot\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}+\frac{\frac{4}{5}\cdot\left(\frac{4}{5}-1\right)\cdot\left(\frac{4}{5}-2\right)}{1.2.3}\cdot\frac{x^{\frac{3}{4}}}{a^{\frac{3}{4}}}\right\}$$

$$=a^{\frac{1}{5}}+\frac{4}{5}\cdot a^{-\frac{1}{20}}.x^{\frac{1}{4}}-\frac{2}{25}a^{-\frac{3}{10}}.x^{\frac{1}{2}}+\frac{4}{125}\cdot a^{-\frac{11}{20}}.x^{\frac{3}{4}}.$$

7. $$1-\frac{x^2}{2}+\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{1.2}\cdot x^4-\frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\cdot\left(\frac{1}{2}-2\right)}{1.2.3}x^6$$
$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}x^8$$
$$=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}.$$

8. $$1-\frac{7a^2}{3}+\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)}{1.2}a^4-\frac{\frac{7}{3}\cdot\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{1.2.3}a^6$$
$$=1-\frac{7a^2}{3}+\frac{14a^4}{9}-\frac{14a^6}{81}.$$

9. $$1-\frac{9x}{4}+\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)}{1.2}(3x)^2-\frac{\frac{3}{4}\cdot\left(\frac{3}{4}-1\right)\cdot\left(\frac{3}{4}-2\right)}{1.2.3}\cdot(3x)^3$$
$$=1-\frac{9x}{4}-\frac{27x^2}{32}-\frac{135x^3}{128}.$$

10. $$x^3\left\{1-\frac{3}{2}\cdot\frac{2y}{3x^2}+\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)}{1.2}\cdot\frac{4y^2}{9x^4}-\frac{\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1.2.3}\cdot\frac{8y^3}{27x^6}\right\}$$
$$=x^3-xy+\frac{y^2}{6x}+\frac{y^3}{54x^3}.$$

11. $$1-\frac{5}{6}x+\frac{\frac{5}{6}\cdot\left(\frac{5}{6}-1\right)}{1.2}x^2-\frac{\frac{5}{6}\left(\frac{5}{6}-1\right)\left(\frac{5}{6}-2\right)}{1.2.3}x^3$$
$$=1-\frac{5}{6}x-\frac{5}{72}x^2-\frac{35}{1296}x^3.$$

12. $$\left(\frac{2x}{3}\right)^{\frac{2}{3}}\left\{1-\frac{2}{3}\cdot\frac{9y}{4x}+\frac{\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)}{1.2}\cdot\frac{81y^2}{16x^2}\right\}$$
$$=\left(\frac{2}{3}\right)^{\frac{2}{3}}.x^{\frac{2}{3}}-\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{1}{3}}y-\frac{3}{8}\left(\frac{3}{2}\right)^{\frac{1}{3}}.x^{-\frac{4}{3}}y^2.$$

CLIII.

1. $1-2a+\frac{(-2)(-2-1)}{1.2}a^2+\frac{(-2)(-2-1)(-2-2)}{1.2.3}a^3+$

$\frac{(-2)(-2-1)(-2-2)(-2-3)}{1.2.3.4}a^4=1-2a+3a^2-4a^3+5a^4.$

2. $1+3x+\frac{(-1)\cdot(-2)}{1.2}(3x)^2-\frac{(-1).(-2).(-3)}{1.2.3}(3x)^3+$

$\frac{(-1).(-2).(-3).(-4)}{1.2.3.4}(3x)^4=1+3x+9x^2+27x^3+81x^4.$

3. $1+x+\frac{(-4).(-5)}{1.2}\cdot\frac{x^2}{16}-\frac{(-4).(-5).(-6)}{1.2.3}\cdot\frac{x^3}{64}$

$=1+x+\frac{5x^2}{8}+\frac{5x^3}{16}.$

4. $1+x+\frac{(-2).(-3)}{1.2}\cdot\frac{x^2}{4}-\frac{(-2).(-3).(-4)}{1.2.3}\cdot\frac{x^3}{8}+$

$\frac{(-2).(-3).(-4).(-5)}{1.2.3.4}\cdot\frac{x^4}{16}=1+x+\frac{3x^2}{4}+\frac{x^3}{2}+\frac{5x^4}{16}.$

5. $a^{-10}-(-5)a^{-12}.2x+\frac{(-5)(-6)}{1.2}a^{-14}.4x^2-\frac{(-5).(-6)(-7)}{1.2.3}.a^{-16}.8x^3$

$+\frac{(-5).(-6).(-7).(-8)}{1.2.3.4}a^{-18}.16x^4$

$=a^{-10}+10a^{-12}x+60a^{-14}x^2+280a^{-16}.x^3+1120a^{-18}x^4.$

6. $a^{-2}-(-6)a^{-\frac{7}{3}}\cdot x^{\frac{1}{3}}+\frac{(-6)\cdot(-7)}{1.2}\cdot a^{-\frac{8}{3}}\cdot x^{\frac{2}{3}}-\frac{(-6)\cdot(-7)\cdot(-8)}{1.2.3}\cdot a^{-3}x$

$=\frac{1}{a^2}+\frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}}+\frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}}+\frac{56x}{a^3}.$

CLIV.

1. $1-\frac{1}{2}x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}x^4+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}x^6$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}x^8$$

$$=1-\frac{x^2}{2}+\frac{3x^4}{8}-\frac{5x^6}{16}+\frac{35x^8}{128}.$$

2. $$1+\frac{3}{2}x^2+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2}x^4-\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3}x^6$$

$$+\frac{\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)\cdot\left(-\frac{9}{2}\right)}{1.2.3.4}x^8$$

$$=1+\frac{3x^2}{2}+\frac{15x^4}{8}+\frac{35x^6}{16}+\frac{315x^8}{128}.$$

3. $$x^{-2}\left\{1-\frac{2}{5}\cdot\frac{z^5}{x^5}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)}{1.2}\cdot\frac{z^{10}}{x^{10}}+\frac{\left(-\frac{2}{5}\right)\cdot\left(-\frac{7}{5}\right)\cdot\left(-\frac{12}{5}\right)}{1.2.3}\frac{z^{15}}{x^{15}}\right\}$$

$$=x^{-2}-\frac{2}{5}x^{-7}z^5+\frac{7}{25}x^{-12}z^{10}-\frac{28}{125}x^{-17}.z^{15}.$$

4. $$1-x+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}4x^2+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}8x^3$$

$$+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{7}{2}\right)}{1.2.3.4}16x^4$$

$$=1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\frac{35x^4}{8}.$$

5. $$a^{-1}\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2}\cdot\frac{x^4}{a^4}\right.$$

$$\left.+\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1.2.3}\cdot\frac{x^6}{a^6}\right\}$$

$$=\frac{1}{a}-\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}-\frac{5x^6}{16a^7}.$$

6. $$a^{-1}\left\{1-\frac{1}{3}\cdot\frac{x^3}{a^3}+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)}{1.2}\cdot\frac{x^6}{a^6}\right.$$

$$\left.+\frac{\left(-\frac{1}{3}\right)\cdot\left(-\frac{4}{3}\right)\cdot\left(-\frac{7}{3}\right)}{1.2.3}\cdot\frac{x^9}{a^9}\right\}=\frac{1}{a}-\frac{x^3}{3a^4}+\frac{2x^6}{9a^7}-\frac{14x^9}{81a^{10}}.$$

CLV.

The rth term of $(a+x)^n$ is $\frac{n.(n-1)\ .\ .\ .\ (n-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}.\ a^{n-r+1}$.

1. The rth term is $\frac{7\cdot 6\ .\ .\ .\ (9-r)}{1\cdot 2\ .\ .\ .\ (r-1)}x^{r-1}$.

2. The rth term is $(-1)^{r-1}\cdot\frac{12.11\ .\ .\ .\ (14-r)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}$.

3. The rth term is $(-1)^{r-1}\cdot\frac{8.7\ .\ .\ .\ (10-r)}{1.2\ .\ .\ .\ (r-1)}a^{9-r}\cdot x^{r-1}$.

4. The rth term is $\frac{9.8\ .\ .\ .\ (11-r)}{1.2\ .\ .\ .\ (r-1)}\cdot(5x)^{10-r}\cdot(2y)^{r-1}$.

5. The rth term is $\frac{(-2)\cdot(-3)\ .\ .\ .\ (-2-r+2)}{1.2\ .\ .\ .\ (r-1)}x^{r-1}$.

$$=(-1)^{r-1}\cdot\frac{2.3\ .\ .\ .\ r}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}=(-1)^{r-1}\cdot r\cdot x^{r-1}.$$

6. The rth term is $\frac{4.5\ .\ .\ .\ (r+2)}{1.2\ .\ .\ .\ (r-1)}\cdot(3x)^{r-1}$

$$=\frac{r\cdot(r+1)\cdot(r+2)}{1.2.3}\cdot(3x)^{r-1}.$$

7. The rth term is $\frac{\frac{1}{2}\cdot\frac{3}{2}\ .\ .\ .\ \left(r-\frac{3}{2}\right)}{1.2\ .\ .\ .\ (r-1)}\cdot x^{r-1}$

$$=\frac{\left\{1.3.5\ \dots\ (2r-3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ \dots\ (r-1)}\cdot x^{r-1}$$

$$=\frac{1.3.5\ \dots\ (2r-3)}{1.2\ \dots\ (r-1)}\cdot\left(\frac{x}{2}\right)^{r-1}.$$

8. The rth term is $(-1)^{r-1}\cdot\dfrac{\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{5}{3}\ \dots\ \left(r-\frac{7}{3}\right)}{1.2\ \dots\ (r-1)}\cdot a^{\frac{1}{3}-r+1}x^{r-1}$

$$=(-1)^{r-1}\cdot\frac{\left\{1.2.5\ \dots\ (3r-7)\right\}\times\left(\frac{1}{3}\right)^{r-1}}{1.2\ \dots\ (r-1)}.a^{\frac{1}{3}}\cdot\left(\frac{x}{a}\right)^{r-1}$$

$$=(-1)^{r-1}\cdot\frac{1.2.5\ \dots\ (3r-7)}{1.2\ \dots\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(\frac{x}{3a}\right)^{r-1}$$

$$=\frac{1.2.5\ \dots\ (3r-7)}{1.2\ \dots\ (r-1)}\cdot a^{\frac{1}{3}}\cdot\left(-\frac{x}{3a}\right)^{r-1}.$$

9. The rth term is $\dfrac{\frac{7}{2}\cdot\left(\frac{7}{2}+1\right)\ \dots\ \left(\frac{7}{2}+r-2\right)}{1.2\ \dots\ (r-1)}\cdot(2x)^{r-1}$

$$=\frac{\left\{7.9\ \dots\ (2r+3)\right\}\times\left(\frac{1}{2}\right)^{r-1}}{1.2\ \dots\ (r-1)}\cdot(2x)^{r-1}$$

$$=\frac{7.9\ \dots\ (2r+3)}{1.2\ \dots\ (r-1)}\cdot x^{r-1}.$$

10. The rth term is $\dfrac{\frac{3}{4}\cdot\left(\frac{3}{4}+1\right)\ \dots\ \left(\frac{3}{4}+r-2\right)}{1.2\ \dots\ (r-1)}.(a^2)^{-\frac{3}{4}-r+1}.(x^2)^{r-1}$

$$=\frac{\{3.7\ \dots\ (4r-5)\}\times\left(\frac{1}{4}\right)^{r-1}}{1.2\ \dots\ (r-1)}.a^{-\frac{3}{2}}.\left(\frac{x^2}{a^2}\right)^{r-1}$$

$$=\frac{3.7\ \dots\ (4r-5)}{1.2\ \dots\ (r-1)}\cdot\frac{a^{-\frac{3}{2}}}{4^{r-1}}\cdot\left(\frac{x}{a}\right)^{2(r-1)}.$$

11. The $(r+1)$th term is $\dfrac{3.4\ .\ .\ .\ (3+r-1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{3.4\ .\ .\ .\ (2+r)}{1.2\ .\ .\ .\ r}.x^r=\frac{(r+1)(r+2)}{2}.x^r.$$

12. The $(r+1)$th term is $\dfrac{\frac{1}{2}\left(\frac{1}{2}+1\right)\ .\ .\ .\ \left(\frac{1}{2}+r-1\right)}{1.2\ .\ .\ .\ r}(4x)^r$

$$=\frac{\{1.3\ .\ .\ .\ (2r-1)\}\times\left(\frac{1}{2}\right)^r}{1.2\ .\ .\ .\ r}.(4x)^r$$

$$=\frac{1.3\ .\ .\ .\ (2r-1)}{1.2\ .\ .\ .\ r}.(2x)^r.$$

13. The $(r+1)$th term is $\dfrac{2r.(2r-1)\ .\ .\ .\ (r+1)}{1.2\ .\ .\ .\ r}x^r$

$$=\frac{2r.(2r-1)\ .\ .\ .\ (r+1).r.(r-1)\ .\ .\ .\ 2.1}{(\underline{|r})^2}.x^r$$

$$=\frac{(2r-1).(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2r.(2r-2)\ .\ .\ .\ 4.2}{\underline{|r}}.x^r$$

$$=\frac{(2r-1)(2r-3)\ .\ .\ .\ 3.1}{\underline{|r}}.\frac{2^r.(\underline{|r})}{\underline{|r}}.x^r$$

$$=\frac{1.3.5\ .\ .\ .\ (2r-1)}{1.2.3\ .\ .\ .\ r}.(2x)^r.$$

14. Coefficient of x^{r+1} in $(1+x)^{n+1}$ is $\dfrac{(n+1.)n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}$

Coefficient of x^r in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ r}$.

Coefficient of x^{r+1} in $(1+x)^n$ is $\dfrac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}$.

Now $\dfrac{(n+1)n(n-1)\ .\ .\ .\ (n-r+1)}{1.2.3\ .\ .\ .\ (r+1)}=\dfrac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}.\dfrac{n+1}{r+1}$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}.\left(1+\frac{n-r}{r+1}\right)$$

$$=\frac{n.(n-1)\ .\ .\ .\ (n-r+1)}{1.2\ .\ .\ .\ r}+\frac{n.(n-1)\ .\ .\ .\ (n-r)}{1.2\ .\ .\ .\ (r+1)}.$$

15. $$\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}.(a)^{-\frac{1}{2}-3}.\left(\frac{1}{x}\right)^3=\frac{5}{16}a^{-\frac{7}{2}}.\frac{1}{x^3}.$$

16. $$\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{1.2.3.4}.(a^2)^{\frac{3}{2}-4}.(-b^2)^4=\frac{3}{128}a^{-5}.b^8.$$

17. $$-\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{1.2.3.4.5.6.7.8}.(a^2)^{\frac{1}{2}-8}.(2x^2)^8$$
$$=-\frac{429}{128}.a^{-15}.x^{16}.$$

18. $$-\frac{m.(m+1)\ .\ .\ .\ (m+8)}{1.2\ .\ .\ .\ 9}.a^{-(m+9)}.b^9.$$

19. $$\frac{\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)\left(\frac{1}{m}-2\right)\ .\ .\ .\ \left(\frac{1}{m}-5\right)}{1.2\ .\ .\ .\ 6}a^{\frac{1}{m}-6}b^6$$
$$=\frac{(1-m)(1-2m)\ .\ .\ .\ (1-5m)}{1.2\ .\ .\ .\ 6.m^6}.a^{\frac{1}{m}-6}.b^6.$$

CLVI.

1. $$\sqrt[3]{31}=\sqrt[3]{(27+4)}=3\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$$
$$=3\left\{1+\frac{1}{3}\cdot\frac{4}{27}+\frac{1}{2}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\frac{16}{729}\right.$$
$$+\frac{1}{6}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\frac{64}{19683}$$
$$\left.+\frac{1}{24}\cdot\frac{1}{3}\cdot\left(\frac{1}{3}-1\right)\cdot\left(\frac{1}{3}-2\right)\cdot\left(\frac{1}{3}-3\right).\frac{256}{531441}+\ .\ .\ .\ .\right\}$$
$$=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\frac{2560}{43046721}+\ .\ .\ .\ .$$
$$=3+\frac{6085724}{43046721}=3{\cdot}1413749\ .\ .\ .\ .$$

2. $\sqrt[7]{108} = \sqrt[7]{(128-20)} = 2\left(1-\frac{5}{32}\right)^{7}$

$$= 2\left\{1 - \frac{1}{7}\cdot\frac{5}{32} + \frac{1}{2}\cdot\frac{1}{7}\left(\frac{1}{7}-1\right)\frac{25}{1024}\right.$$

$$-\frac{1}{6}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)\cdot\frac{125}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{7}\cdot\left(\frac{1}{7}-1\right)\cdot\left(\frac{1}{7}-2\right)\cdot\left(\frac{1}{7}-3\right)\cdot\frac{625}{1048576} - \ldots.\right\}$$

$$= 2\left\{1 - \frac{5}{224} - \frac{75}{50176} - \frac{1625}{11239424} - \frac{40625}{2517630976}\ \ldots.\right\}$$

$$= 2\left\{1 - \frac{56197120 + 3763200 + 364000 + 40625}{2517630976}\right\}$$

$$= 2\left(1 - \frac{60364945}{2517630976}\right) = \frac{2457266031}{1258815488} = 1{\cdot}95204\ \ldots.$$

3. $\sqrt[5]{260} = \sqrt[5]{(243+17)} = 3\left(1+\frac{17}{243}\right)^{\frac{1}{5}}$

$$= 3\left\{1 + \frac{1}{5}\cdot\frac{17}{243} + \frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{289}{59049}\right.$$

$$\left.+\frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{4913}{14348907}\right\}$$

$$= 3.\left\{1 + \frac{17}{1215} - \frac{578}{1476225} + \frac{29478}{1793613375}\ \ldots.\right\}$$

$$= 3\left\{1 + \frac{25095825 - 702270 + 29478}{1793613375}\ \ldots.\right\}$$

$$= 3.\left\{1 + \frac{24423033}{1793613375}\right\} = 3 + \frac{24423033}{597871125} = 3{\cdot}04084\ \ldots.$$

4. $\sqrt[5]{31} = \sqrt[5]{(32-1)} = 2\left(1-\frac{1}{32}\right)^{\frac{1}{5}} = 2\left\{1 - \frac{1}{5}\cdot\frac{1}{32}\right.$

$$+\frac{1}{2}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\frac{1}{1024} - \frac{1}{6}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\frac{1}{32768}$$

$$\left.+\frac{1}{24}\cdot\frac{1}{5}\cdot\left(\frac{1}{5}-1\right)\cdot\left(\frac{1}{5}-2\right)\cdot\left(\frac{1}{5}-3\right)\cdot\frac{1}{1048576}\cdots\right\}$$

$$=2\left\{1-\frac{1}{160}-\frac{1}{12800}-\frac{3}{4096000}-\frac{21}{655360000}\ldots\right\}$$

$$=2\left\{1-\frac{4096000+51200+480+21}{655360000}\ldots\right\}$$

$$=2\left\{1-\frac{4147701}{655360000}\right\}=1\cdot98734\ldots$$

CLVII.

1. $$\begin{array}{r}23561\\42513\\645325\\\hline 1045032\end{array}$$

2. $$\begin{array}{r}3074852\\4635628\\1247153\\\hline 10070344\end{array}$$

3. $$\begin{array}{r}358423\\267862\\\hline 80451\end{array}$$

4. $$\begin{array}{r}211010\\124321\\\hline 31134\end{array}$$

5. $$\begin{array}{r}57264\\675\\\hline 354604\\513354\\434070\\\hline 51117344\end{array}$$

6. $$\begin{array}{r}1456\\6451\\\hline 1456\\6523\\11312\\13101\\\hline 14332216\end{array}$$

7. $$\begin{array}{l}5\overline{)243012}\\\quad 31450 \text{ rem. } 2.\end{array}$$

8. $$\begin{array}{l}6\overline{)3756025}\\\quad 522256 \text{ rem. } 1.\end{array}$$

9. $$\begin{array}{r|r}& 25|40|05|44(4112\\& 24\\\hline 121 & 140\\& 121\\\hline 1221 & 1505\\& 1221\\\hline 12222 & 24444\\& 24444\\\hline\end{array}$$

10. $$\begin{array}{r|r}& 5|68|98|t1(2437\\& 4\\\hline 44 & 168\\& 154\\\hline 483 & 1498\\& 1209\\\hline 4867 & 28et1\\& 28et1\\\hline\end{array}$$

CLVIII.

1.
$$\begin{array}{r|l} 7 & 1828 \\ 7 & 261-1 \\ 7 & 37-2 \\ 7 & 5-2 \\ & 0-5 \end{array}$$

2.
$$\begin{array}{r|l} 6 & 1820 \\ 6 & 303-2 \\ 6 & 50-3 \\ 6 & 8-2 \\ 6 & 1-2 \\ & 0-1 \end{array}$$

3.
$$\begin{array}{r|l} 12 & 43751 \\ 12 & 3645-e \\ 12 & 303-9 \\ 12 & 25-3 \\ 12 & 2-1 \\ & 0-2 \end{array}$$

4.
$$\begin{array}{r|l} 5 & 3700 \\ 5 & 740-0 \\ 5 & 148-0 \\ 5 & 29-3 \\ 5 & 5-4 \\ 5 & 1-0 \\ & 0-1 \end{array}$$

5.
$$\begin{array}{r|l} 2 & 7631 \\ 2 & 3815-1 \\ 2 & 1907-1 \\ 2 & 953-1 \\ 2 & 476-1 \\ 2 & 238-0 \\ 2 & 119-0 \\ 2 & 59-1 \\ 2 & 29-1 \\ 2 & 14-1 \\ 2 & 7-0 \\ 2 & 3-1 \\ 2 & 1-1 \\ & 0-1 \end{array}$$

6.
$$\begin{array}{r|l} 12 & 215855 \\ 12 & 17987-e \\ 12 & 1498-e \\ 12 & 124-t \\ 12 & 10-4 \\ & 0-t \end{array}$$

7.
$$\begin{array}{r|l} 7 & 790158 \\ 7 & 112879-5 \\ 7 & 16125-4 \\ 7 & 2303-4 \\ 7 & 329-0 \\ 7 & 47-0 \\ 7 & 6-5 \\ & 0-6 \end{array}$$

8.
$$\begin{array}{r|l} 4 & 34002 \\ 4 & 4334-1 \\ 4 & 1043-2 \\ 4 & 122-0 \\ 4 & 14-1 \\ 4 & 2-1 \\ & 0-2 \end{array}$$

9.
$$\begin{array}{r|r}
12 & \underline{8978} \\
12 & \underline{816-2} \\
12 & \underline{75-1} \\
12 & \underline{6-t} \\
 & 0-6
\end{array}$$

10.
$$\begin{array}{r|r}
12 & \underline{3256} \\
12 & \underline{166-4} \\
12 & \underline{11-1} \\
 & 0-8
\end{array}$$

11.
$$\begin{array}{r|r}
8 & \underline{37704} \\
8 & \underline{4311-5} \\
8 & \underline{480-1} \\
8 & \underline{54-4} \\
8 & \underline{6-1} \\
 & 0-6
\end{array}$$

12.
$$\begin{array}{r|r}
4 & \underline{5056} \\
4 & \underline{1165-0} \\
4 & \underline{214-3} \\
4 & \underline{36-1} \\
4 & \underline{6-3} \\
4 & \underline{1-2} \\
 & 0-1
\end{array}$$

13.
$$\begin{array}{r|r}
7 & \underline{654321} \\
7 & \underline{e0738-5} \\
7 & \underline{16e3e-3} \\
7 & \underline{2858-3} \\
7 & \underline{478-0} \\
7 & \underline{7e-3} \\
7 & \underline{11-4} \\
7 & \underline{1-6} \\
 & 0-1
\end{array}$$

14.
$$\begin{array}{r|r}
11 & \underline{2304} \\
11 & \underline{104-t} \\
11 & \underline{2-7} \\
 & 0-2
\end{array}$$

CLIX.

1.
$$\left.\begin{array}{l}
\frac{25}{36}\times 6=4+\frac{6}{36} \\
\frac{6}{36}\times 6=1+0
\end{array}\right\}\therefore\ \cdot 41 \text{ is the result.}$$

2.
$$\frac{3}{11}\times 7=1+\frac{10}{11};\ \frac{10}{11}\times 7=6+\frac{4}{11};\ \frac{4}{11}\times 7=2+\frac{6}{11};$$

$$\frac{6}{11}\times 7=3+\frac{9}{11};\ \frac{9}{11}\times 7=5+\frac{8}{11};\ \frac{8}{11}\times 7=5+\frac{1}{11};$$

$$\frac{1}{11}\times 7=1+\frac{7}{11};\ \frac{7}{11}\times 7=4+\frac{5}{11};\ \frac{5}{11}\times 7=3+\frac{2}{11};$$

$$\frac{2}{11}\times 7=1+\frac{3}{11};\ \frac{3}{11}\times 7=1+\frac{10}{11}$$

∴ result is $\cdot\dot{1}6235504\dot{3}$.

3. $$\begin{array}{r|l} 9 & 23 \\ \hline 9 & 2-5 \\ \hline & 0-2 \end{array} \qquad \begin{array}{r} \cdot125 \\ 9 \\ \hline 1\cdot125 \\ 9 \\ \hline 1\cdot125 \end{array}$$

∴ result is $25\cdot\dot{1}$.

4. $$\begin{array}{r|l} 6 & 1820 \\ \hline 6 & 303-2 \\ \hline 6 & 50-3 \\ \hline 6 & 8-2 \\ \hline 6 & 1-2 \\ \hline & 0-1 \end{array} \qquad \begin{array}{r} \cdot3375 \\ 6 \\ \hline 2\cdot0250 \\ 6 \\ \hline 0\cdot1500 \\ 6 \\ \hline 0\cdot9000 \\ 6 \\ \hline 5\cdot4000 \\ 6 \\ \hline 2\cdot4000 \end{array}$$

∴ result is $12232\cdot2005\dot{2}$.

5. $2x^5+x^4+2x^3+5x^2+4x+2-17486=0$;

$(x-6)(2x^4+13x^3+80x^2+485x+2914)=0$;

∴ $x=6$. Hence the scale is senary.

6. $x^6+7x^5+4x^4+6x^3+3x^2+5-511173=0$;

$(x-8)(x^5+15x^4+124x^3+998x^2+7987x+63896)=0$;

∴ $x=8$. Hence the scale is octenary.

7. (1.) Let N be the number, and suppose

$$N = a.10^n + b.10^{n-1} + \ldots + m.100 + p.10 + q.$$

Then $N = a(10^n - 1) + b(10^{n-1} - 1) + \ldots + m(100 - 1)$
$+ p(10 - 1) + (a + b + \ldots + m + p + q)$.

Now all the expressions $10^n - 1, 10^{n-1} - 1 \ldots 10^2 - 1, 10 - 1$ are divisible by $10 - 1$, or 9, and therefore by 3.

Hence N is divisible by 3, if $a + b + \ldots + m + p + q$ be divisible by 3.

(2.) Let $N = a.10^n + b.10^{n-1} + \ldots + m.100 + p.10 + q$.

Now 100 and all its multiples are divisible by 4,

$\therefore N$ is divisible by 4 if $10p + q$ be divisible by 4.

(3.) Let $N = a.10^n + b.10^{n-1} + \ldots + m.100 + p.10 + q$.

Now 1000 and all its multiples are divisible by 8,

$\therefore N$ is divisible by 8 if $100m + 10p + q$ be divisible by 8.

(4.) Let $N = a.10^n + b.10^{n-1} + \ldots + m.100 + p.10 + q$.

Now 10 and all its multiples are divisible by 5,

$\therefore N$ is divisible by 5 if $q = 5$ or $q = 0$.

(5.) Let N be the number, $p_n, p_{n-1}, \ldots p_4, p_3, p_2, p_1$, the digits.

Then $N = p_1 + 10.p_2 + 100p_3 + 1000p_4 + \ldots + 10^{n-1}.p_{n-1} + 10^n p^n$
$= p_1 - p_2 + p_3 - p_4 + \ldots + (-1)^n.p_n$
$+ p_2(10 + 1) + p_3(10^2 - 1) + p_4(10^3 + 1) + \ldots + p_n\{10^n - (-1)^n\}$.

Now $10 + 1, 10^2 - 1 \ldots$ are all divisible by $10 + 1$, or 11,

$\therefore N$ is divisible by 11, if $(p_1 + p_2 + \ldots) - (p_2 + p_4 \ldots)$ be divisible by 11.

8. Let $N = a.r^n + b.r^{n-1} + \ldots + m.r^2 + pr + q$,
then $n = a + rb + \ldots + m.r^{n-2} + p.r^{n-1} + q.r^n$.

Then $N - n = a(r^n - 1) + b.(r^{n-1} - r) + \ldots - m(r^{n-2} - r^2)$
$- p(r^{n-1} - r) - q(r^n - 1)$, and each of the factors $r^n - 1$,
$r^{n-1} - r \ldots r^{n-2} - r^2, r^{n-1} - r, r^n - 1$ is divisible by $r - 1$;
$\therefore N - n$ is divisible by $r - 1$.

CLX.

1. $$\begin{array}{r} \overline{3}\cdot1651553 \\ \overline{4}\cdot7505855 \\ 6\cdot6879746 \\ \overline{2}\cdot6150026 \\ \hline \overline{1}\cdot2187180 \end{array}$$

2. $$\begin{array}{r} \overline{4}\cdot6843785 \\ \overline{5}\cdot6650657 \\ 3\cdot8905196 \\ \overline{3}\cdot4675284 \\ \hline \overline{7}\cdot7074922 \end{array}$$

3. $$\begin{array}{r} 2\cdot5324716 \\ 3\cdot6650657 \\ \overline{5}\cdot8905196 \\ \cdot3156215 \\ \hline 2\cdot4036784 \end{array}$$

4. $$\begin{array}{r} 2\cdot483269 \\ \overline{3}\cdot742891 \\ \hline 4\cdot740378 \end{array}$$

5. $$\begin{array}{r} \overline{2}\cdot352678 \\ \overline{5}\cdot428619 \\ \hline 2\cdot924059 \end{array}$$

6. $$\begin{array}{r} \overline{5}\cdot349162 \\ \overline{3}\cdot624329 \\ \hline \overline{3}\cdot724833 \end{array}$$

7. $$\begin{array}{r} \overline{2}\cdot4596721 \\ 3 \\ \hline \overline{5}\cdot3790163 \end{array}$$

8. $$\begin{array}{r} \overline{7}\cdot429683 \\ 6 \\ \hline \overline{40}\cdot578098 \end{array}$$

9. $$\begin{array}{r} \overline{9}\cdot2843617 \\ 7 \\ \hline \overline{62}\cdot9905319 \end{array}$$

10. $$\begin{array}{r|l} 3 & \overline{6}\cdot3725409 \\ \cline{2-2} & \overline{2}\cdot1241803 \end{array}$$

11. $$\begin{array}{r|l} 6 & \overline{14}\cdot432962 \\ \cline{2-2} & \overline{3}\cdot738827 \end{array}$$

12. $$\begin{array}{r|l} 9 & \overline{4}\cdot53627188 \\ \cline{2-2} & \overline{1}\cdot61514132 \end{array}$$

CLXI.

1. $\text{Log } 128 = \log 2^7 = 7 \log 2 = 2\cdot1072100.$

$$\text{Log } 125 = \log \frac{1000}{8} = \log 1000 - \log 8 = 3 - \log 2^3$$

$$= 3 - 3 \log 2 = 3 - \cdot9030900 = 2\cdot0969100.$$

$$\text{Log } 2500 = \log \frac{10000}{4} = \log 10000 - \log 4 = 4 - 2 \log 2$$

$$= 4 - \cdot6020600 = 3\cdot3979400.$$

2. $$\text{Log } 50 = \log \frac{100}{2} = \log 100 - \log 2 = 2 - \cdot3010300 = 1\cdot6989700.$$

$$\text{Log } \cdot005 = \log \frac{5}{1000} = \log 10 - \log 2 - 3 = -\log 2 - 2 = \overline{3}\cdot6989700.$$

$$\text{Log } 196 = \log (49 \times 4) = 2 \log 7 + 2 \log 2 = 2\cdot2922560.$$

3. $\text{Log } 6 = \log 3 + \log 2 = {\cdot}7781513.$

$\text{Log } 27 = 3 \log 3 = 1{\cdot}4313639.$

$\text{Log } 54 = \log (27 \times 2) = 3 \log 3 + \log 2 = 1{\cdot}7323939.$

$\text{Log } 576 = \log (9 \times 64) = 2 \log 3 + 6 \log 2 = 2{\cdot}7604226.$

4. $\text{Log } 60 = \log (2 \times 3 \times 10) = \log 2 + \log 3 + \log 10 = 1{\cdot}7781513.$

$\text{Log } {\cdot}03 = \log \frac{3}{100} = \log 3 - 2 = {\cdot}4771213 - 2 = \bar{2}{\cdot}4771213.$

$\text{Log } 1{\cdot}05 = \log \frac{105}{100} = \log \frac{21}{20} = \log 3 + \log 7 - \log 2 - 1 = {\cdot}0211893.$

$\text{Log } {\cdot}0000432 = \log \frac{16 \times 27}{10000000} = 4 \log 2 + 3 \log 3 - 7 = \bar{5}{\cdot}6354839.$

5. $\text{Log } {\cdot}00075 = \log 75 - 5 = \log 3 + \log 25 - 5 = \log \left(\frac{18}{2}\right)^{\frac{1}{2}} + \log 25 - 5$

$= \frac{1}{2} \{\log 18 - \log 2\} + \log 100 - \log 4 - 5.$

$= \frac{1}{2} \{1{\cdot}2552725 - {\cdot}3010300\} + 2 - {\cdot}6020600 - 5$

$= {\cdot}4771213 - {\cdot}6020600 - 3 = \bar{4}{\cdot}8750613.$

$\text{Log } 31{\cdot}5 = \log (21 \times 3 \times 5) - 1 = \log 21 + \log 3 + 1 - \log 2 - 1.$

$= \log 21 + \frac{1}{2} \{\log 18 - \log 2\} - \log 2$

$= 1{\cdot}3222193 + {\cdot}4771212 - {\cdot}3010300 = 1{\cdot}4983105.$

6. $\text{Log } 2 = \log \frac{10}{5} = 1 - \log 5 = {\cdot}3010300$

$\text{Log } {\cdot}064 = \log. \frac{2^6}{1000} = 6 \log 2 - 3 = 6 - 6 \log 5 - 3 = \bar{2}{\cdot}8061800$

$\text{Log } \left\{\frac{2^{60}}{5^{20}}\right\}^{\frac{1}{14}} = \frac{1}{14} (60 \log 2 - 20 \log 5)$

$= \frac{1}{7} (30 - 30 \log. 5 - 10 \log 5) = \frac{1}{7} (30 - 27{\cdot}9583000)$

$= \frac{1}{7} (2{\cdot}0412000) = {\cdot}2916000.$

7. $\text{Log } 5 = \log \frac{10}{2} = 1 - {\cdot}3010300 = {\cdot}6989700$

$$\log {\cdot}125 = \log \frac{5^3}{1000} = 3 \log 5 - 3 = 2{\cdot}0969100 - 3 = \bar{1}{\cdot}0969100$$

$$\text{Log } \left(\frac{5^{90}}{2^{40}}\right)^{\frac{1}{15}} = \log 5^{\frac{90}{15}} - \log 2^{\frac{40}{15}} = \log 5^6 - \log 2^{\frac{8}{3}}$$

$$= 6 \log 5 - \frac{8}{3} \log 2 = 6 \ (\log 10 - \log 2) - \frac{8}{3} \log 2$$

$$= 4{\cdot}1938200 - {\cdot}8027467 = 3{\cdot}3910733.$$

8. $-2, 0, 2$; $1, 0, -1$.

9. 1593 is greater than 10^3 and less than 10^4; characteristic 3; 1593 is greater than 12^2 and less than 12^3; characteristic 2.

10. $\frac{4^{3y}}{2^{4}_{y}} = 8$; $\frac{2^{6y}}{2^{4y}} = 2^3$; $2^{2y} = 2^3$; $2y = 3$, etc.

11. (*a*) $\log 2 = \frac{1}{2} \log 4 = {\cdot}3010300.$

$$\text{Log } 25 = \log 100 - \log 4 = 2 - {\cdot}6020600 = 1{\cdot}3979400$$

$$\text{Log } 83{\cdot}2 = \log (8 \times 1{\cdot}04) = \frac{3}{2} \log 4 + \log 1{\cdot}04$$

$$= {\cdot}9030900 + {\cdot}0170333 = 1{\cdot}9201233.$$

$$\text{Log } ({\cdot}625)^{\frac{1}{100}} = \frac{1}{100} \left\{ \log 625 - \log 1000 \right\} = \frac{1}{100} \left\{ 2 \log 25 - 3 \right\}$$

$$= \frac{1}{100} \left\{ 2 \log 100 - 2 \log 4 - 3 \right\} = \frac{1}{100} \left\{ 4 - 1{\cdot}2041200 - 3 \right\}$$

$$= -{\cdot}0020412 = \bar{1}{\cdot}9979588.$$

(*b*) $\text{Log } (1{\cdot}04)^{6000} = 6000 \log 1{\cdot}04 = 6000 \times {\cdot}0170333$

$= 102{\cdot}1998000$; $\therefore$ number of digits is 103.

12. (*a*) $\text{Log}\ 5 = \frac{1}{2}\log 25 = \cdot 6989700$

$\text{Log}\ 4 = 2 - \log 25 = \cdot 6020600$

$\text{Log}\ 51\cdot 5 = \log 5 + \log 10\cdot 3 = \cdot 6989700 + 1\cdot 0128372 = 1\cdot 7118072$

$\text{Log}\ (\cdot 064)^{\frac{1}{100}} = \frac{1}{100}\left\{\log 64 - \log 1000\right\} = \frac{1}{100}\left\{3\log 4 - 3\right\}$

$= \frac{1}{100}\left\{1\cdot 8061800 - 3\right\} = -\cdot 0119382 = \bar{1}\cdot 9880618.$

(*b*) $\text{Log}\ (1\cdot 03)^{600} = 600\log 1\cdot 03 = 600 \times \cdot 0128372$

$= 7\cdot 7023200$; $\therefore$ number of digits is 8.

13. $\text{Log}\ 7623 = 2\log 3 + 2\log 11 + \log 7$

$= \cdot 9542426 + 2\cdot 0827854 + \cdot 8450980 = 3\cdot 8821260$

$\text{Log}\ \frac{77}{300} = \log 7 + \log 11 - \log 3 - \log 100$

$= \cdot 8450980 + 1\cdot 0413927 - \cdot 4771213 - 2 = \bar{1}\cdot 4093694$

$\text{Log}\ \frac{3}{539} = \log 3 - \log 11 - 2\log 7$

$= \cdot 4771213 - 1\cdot 0413927 - 1\cdot 6901960 = \bar{3}\cdot 7455326.$

14. (1.) $x\log 4096 = \log 8 - x\log 64$; $4x\log 8 = \log 8 - 2x\log 8$;

$4x = 1 - 2x$; $6x = 1$; $x = \frac{1}{6}$.

(2.) $(2\cdot 5)^x = 6\cdot 25 = (2\cdot 5)^2$; $\therefore x = 2$.

(3.) $(ab)^x = m$; $x\log(ab) = \log m$

$\therefore x = \dfrac{\log m}{\log a + \log b}$.

(4.) $x(m\log a + 2\log b) = \log c$, etc.

(5.) $3x\log a + (4-x)\log b = (2x-1)\log c$

$x(3\log a - \log b - 2\log c) = -4\log b - \log c$, etc.

(6.) $x(\log a + m\log b) = \log c - 3x\log c$

$x(\log a + m\log b + 3\log c) = \log c$, etc.

CLXII.

1. $P(1+r)^n=2P$; $(1+r)^n=2$; $\left(1+\frac{4}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 104-\log 100}=\frac{{\cdot}3010300}{{\cdot}0170333}=17{\cdot}6\ .\ .\ .$$

2. $P(1+r)^n=2P$; $\left(1+\frac{3}{100}\right)^n=2$;

$$\therefore n=\frac{\log 2}{\log 1{\cdot}03}=\frac{{\cdot}3010300}{{\cdot}0128372}=23{\cdot}4\ .\ .\ .$$

3. $\left(1+\frac{10}{100}\right)^n=2$; $n=\frac{\log 2}{\log 1{\cdot}1}=\frac{{\cdot}3010300}{{\cdot}0413927}=7{\cdot}2725\ .\ .\ .$

4. $\left(1+\frac{5}{100}\right)^n=3$; $n=\frac{\log 3}{\log 1{\cdot}05}=\frac{{\cdot}4771213}{{\cdot}0211893}=22{\cdot}5$ nearly.

5. $P(1+r)^n=2P$; $\therefore n=\frac{\log 2}{\log (1+r)}$

$$P(1+2r)^m=2P\ ;\ \therefore m=\frac{\log 2}{\log (1+2r)}$$

$$\therefore \frac{m}{n}=\frac{\log (1+r)}{\log (1+2r)}.$$

Now $1+2r$ is less than $(1+r)^2$

$$\therefore \frac{m}{n} \text{ is greater than } \frac{\log (1+r)}{\log (1+r)^2},\ \text{or},\ \frac{\log (1+r)}{2\log (1+r)},\ \text{or},\ \frac{1}{2}.$$

6. $1000(1+r)^n=1800$; $\left(1+\frac{5}{100}\right)^n=1{\cdot}8$

$$\therefore n\ \log\ 1{\cdot}05=\log 1{\cdot}8\ ;\ n=\frac{{\cdot}2552725}{{\cdot}0211893}=12 \text{ nearly.}$$

7. $P(1+r)^{2n}=2P$; $\left(1+\frac{3}{100}\right)^{2n}=2$

$$\therefore 2n=\frac{\log 2}{\log 1{\cdot}03}=\frac{{\cdot}3010300}{{\cdot}0128372}=23{\cdot}449\ .\ .\ .$$

$$\therefore n=11{\cdot}724\ .\ .\ .\ .$$

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[*Specimen Page, No.* 1.]

Give every man thy ear, but few thy voice;
Take each man's censure, but reserve thy judgment.
Costly thy habit as thy purse can buy,
But not express'd in fancy; rich, not gaudy;
For the apparel oft proclaims the man;
And they in France of the best rank and station
Are of a most select and generous chief in that.
Neither a borrower nor a lender be;
For loan oft loses both itself and friend,
And borrowing dulls the edge of husbandry.
This above all: to thine ownself be true,

---

68 *Give every man thine ear.* For a good listener is generally thought by the willing speaker to be a man of sound judgment. "Mr. Canning," says Sir E. Bulwer, "would often make a kind of lounging tour of the House, listening to the tone of the observations which the previous debate had excited; so that at last, when he rose to speak, he seemed to a large part of his audience to be merely giving a more striking form to their own thoughts."

71 *Express'd in fancy.* Not marked or singular in device; but with a quiet costliness suggestive of habitual self-respect.

74 *A most select and generous chief.* Are of a most noble device in this—the 'chief' being the upper part of a heraldic shield. The passage is strangely misunderstood and even altered by Delius, Elze, and other editors. As regards the metre, the three first syllables of the line must be pronounced rapidly in the time of one, as in Macbeth, i. 5, we have:

> "And take my milk for gall, you murdering ministers."

76 *Loses itself and friend.* Who ever loves the creditor whom he cannot pay?

77 *Dulls the edge of husbandry.* Takes the edge off economy. Money borrowed, whether by individuals or nations, represents no saving or self-denial, and is therefore lightly parted with.

78 *To thine ownself be true.* As you inwardly resolve, so do: then faithfulness to others as well as yourself becomes the habit of your soul. So Wordsworth (v. 49) speaks of the same steadfastness in—

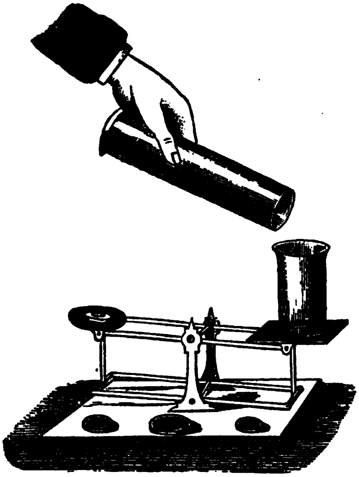
> "The generous spirit who when brought
> Amongst the tasks of real life, has wrought
> Upon the plan that pleased his childish thought."

And, in an equally noble style, an Eastern sage has said, "There is one way to gladden those whom you love: if one is not upright when retired into himself, never will he bring rejoicing to those who are near him."

[*Select Plays of Shakspere. See page* 4.]

In Fig. 16 is represented a very pretty experiment, showing that this gas is heavier than air. First, balance a jar with a weight. I say *balance* a jar. Is that exactly correct? Is there not some thing in the jar? "No," you will perhaps say, "it is empty." But think a moment. That jar is full of something, and that something has weight. It is full of air. We have balanced, then, a jar full of air. Now if, as represented, carbonic acid gas be poured into the jar on the scales, the jar will descend and the weight will rise. Why? Because there is now a gas in the jar that is heavier than air.

Fig. 16.



If you have a jar filled with this gas, you can take it out with a little bucket, as seen in Fig. 17. As you take one bucketful after another out, it can be poured away as water; and air will take the place of the gas as fast as it is removed.

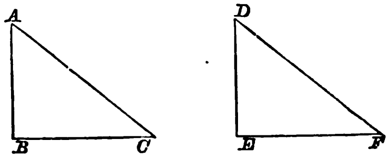
If a soap-bubble fall into a jar of carbonic acid gas, it will not go to the bottom as it would if the jar were full of air. It will descend a little into the jar, and then ascend and remain in its open mouth. Why is this? The air that is blown into the bubble is lighter than the gas in the jar,

[*Easy Introduction to* Chemistry. See page 11.]

[*Specimen Page, No.* 3.]

Proposition B. Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.



In $\triangle$s ABC, DEF,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $BC = EF$.

Then must $AB = DE$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

For if $\triangle DEF$ be applied to $\triangle ABC$, so that E coincides with B, and EF falls on BC;

then $\because EF = BC$, $\therefore F$ will coincide with C;

and $\because \angle DEF = \angle ABC$, $\therefore ED$ will fall on BA;

$\therefore D$ will fall on BA or BA produced.

Again, $\because \angle DFE = \angle ACB$, $\therefore FD$ will fall on CA;

$\therefore D$ will fall on CA or CA produced.

$\therefore D$ must coincide with A, the only pt. common to BA and CA.

$\therefore DE$ will coincide with and $\therefore$ is equal to AB,

and DF .... AC,

and $\angle EDF$ .... $\angle BAC$;

and $\therefore$ the triangles are equal in all respects. Q. E. D.

Cor. Hence, by a process like that in Prop. A, we can prove the following theorem :

If two angles of a triangle be equal, the sides which subtend them are also equal. (Eucl. I. 6.)

[*Elements of Geometry.* *See page* 9.]

thus: if the articles had cost £1 each, the total cost would have been £2478;

∴ as they cost $\frac{1}{6}$ of £1 each, the cost will be £$\frac{2478}{6}$, or £413.

The process may be written thus:

3*s*. 4*d*. is $\frac{1}{6}$ of £1	£2478	= cost of the articles at £1 each.
	£413	= cost .... at 3*s*. 4*d*. ...

Ex. (2). Find the cost of 2897 articles at £2. 12*s*. 9*d*. each.

£2 is 2 × £1	2897 . 0 . 0	= cost at £1 each.
10*s*. is $\frac{1}{2}$ of £1	5794 . 0 . 0	= ...... £2 ....
2*s*. is $\frac{1}{5}$ of 10*s*.	1448 . 10 . 0	= ...... 10*s*.....
8*d*. is $\frac{1}{3}$ of 2*s*.	289 . 14 . 0	= ...... 2*s*. ....
1*d*. is $\frac{1}{8}$ of 8*d*.	96 . 11 . 4	= ...... 8*d*. ....
	12 . 1 . 5	= ...... 1*d*. ....
	£7640 . 16 . 9	= ...... £2. 12*s*. 9*d*. each.

NOTE.—A shorter method would be to take the parts thus:

10*s*. = $\frac{1}{2}$ of £1; 2*s*. 6*d*. = $\frac{1}{4}$ of 10*s*.; 3*d*. = $\frac{1}{10}$ of 2*s*. 6*d*.

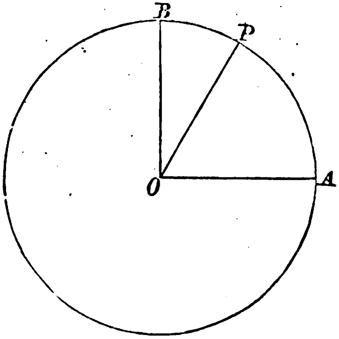
Ex. (3). Find the cost of 425 articles at £2. 18*s*. 4*d*. each.

Since £2. 18*s*. 4*d*. is the difference between £3 and 1*s*. 8*d*. (which is $\frac{1}{12}$ of £1), the shortest course is to find the cost at £3 each, and to *subtract from it* the cost at 1*s*. 8*d*. each, thus:

	£ . *s*. . *d*.	
£3 is 3×£1	425 . 0 . 0	= cost at £1 each.
1*s*. 8*d*. is $\frac{1}{12}$ of £1	1275 . 0 . 0	= ...... £3 ....
	35 . 8 . 4	= ...... 1*s*. 8*d*. each.
	£1239 . 11 . 8	= ...... £2. 18*s*. 4*d*. each.

[*Specimen Page, No.* 5.]

28. ***To shew that the angle subtended at the centre of a circle by an arc equal to the radius of the circle is the same for all circles.***



Let O be the centre of a circle, whose radius is r;

AB the arc of a quadrant, and therefore AOB a right angle;

AP an arc equal to the radius AO.

Then, $AP=r$ and $AB=\frac{\pi r}{2}$. (Art. 14.)

Now, by Euc. VI. 33,

$$\frac{\text{angle } AOP}{\text{angle } AOB}=\frac{\text{arc } AP}{\text{arc } AB},$$

or,

$$\frac{\text{angle } AOP}{\text{a right angle}}=\frac{r}{\frac{\pi r}{2}}$$

$$=\frac{2r}{\pi r}$$

$$=\frac{2}{\pi}.$$

Hence
$$\text{angle } AOP=\frac{2 \text{ right angles}}{\pi}.$$

Thus the magnitude of the angle AOP is independent of r and is therefore the same for all circles.

[*Trigonometry. See page* 10.]

89. Case II. The next case in point of simplicity is that in which four terms can be so arranged, that the first two have a common factor and the last two have a common factor.

Thus

$$\begin{aligned} x^2+ax+bx+ab &= (x^2+ax)+(bx+ab) \\ &= x(x+a)+b(x+a) \\ &= (x+b)(x+a). \end{aligned}$$

Again

$$\begin{aligned} ac-ad-bc+bd &= (ac-ad)-(bc-bd) \\ &= a(c-d)-b(c-d) \\ &= (a-b)(c-d). \end{aligned}$$

Examples.—XVIII.

Resolve into factors:

1. $x^2-ax-bx+ab.$
2. $ab+ax-bx-x^2.$
3. $bc+by-cy-y^2.$
4. $bm+mn+ab+an.$
5. $abx^2-axy+bxy-y^2.$
6. $abx-aby+cdx-cdy.$
7. $cdx^2+dmxy-cnxy-mny^2.$
8. $abcx-b^2dx-acdy+bd^2y.$

90. Before reading the Articles that follow the student is advised to turn back to Art. 56, and to observe the manner in which the operation of multiplying a binomial by a binomial produces a *trinomial* in the Examples there given. He will then be prepared to expect that in certain cases a *trinomial can be resolved into two binomial factors,* examples of which we shall now give.

91. Case III. To find the factors of

$$x^2+7x+12.$$

Our object is to find two numbers whose product is 12,
and whose sum is 7.

These will evidently be 4 and 3,

$$\therefore\ x^2+7x+12=(x+4)(x+3).$$

Again, to find the factors of

$$x^2+5bx+6b^2.$$

Our object is to find two numbers whose product is $6b^2$,
and whose sum is $5b$.

These will clearly be $3b$ and $2b$,

$$\therefore\ x^2+5bx+6b^2=(x+3b)(x+2b).$$

Praetexta et trabeae, fasces, lectica, tribunal.
Quid, si vidisset praetorem curribus altis
Exstantem et medio sublimem in pulvere Circi,
In tunica Jovis, et pictae Sarrana ferentem
Ex humeris aulaea togae, magnaeque coronae
Tantum orbem, quanto cervix non sufficit ulla?
Quippe tenet sudans hanc publicus, et, sibi Consul
Ne placeat, curru servus portatur eodem.
Da nunc et volucrem, sceptro quae surgit eburno,
Illinc cornicines, hinc praecedentia longi
Agminis officia et niveos ad fraena Quirites,
Defossa in loculis quos sportula fecit amicos.
Tum quoque materiam risus invenit ad omnes
Occursus hominum, cujus prudentia monstrat
Summos posse viros et magna exempla daturos
Vervecum in patria crassoque sub aere nasci.

35] These details are mentioned not as more ridiculous in themselves than anything Democritus had seen in Greece, but because Democritus regarded all human life as a farce, and at Rome the farce was more elaborate. *Lectica* refers to the procession of clients who accompanied it; *tribunal* to the display of empty eloquence before it.

36, sqq.] "What would he have said of the praetor's triumphal procession from the Capitol to the Circus?" The triumphal dress suggests the idea of triumph, and this *consul* (inf. 41).

38 **tunica Jovis**] Whom he personated; hence the eagle on his sceptre. The tunic was so costly that it was not till the third century that a private person possessed one of his own, even the emperors when they triumphed supplied themselves from the treasury of the Capitol or of the Palace.

— **Sarrana**] From the unhellenized form of Tyrus.

39 **aulaea**] A whole stage-curtain of a toga.

41 **Quippe**] "No head could support it: why it makes the slave sweat to hold it up."

44 **longi agminis officia**] There is no more difference between this and longa agmina officiosorum, than between 'a high-spirited nobleman on a long-tailed horse,' and 'a long-tailed nobleman on a high-spirited horse.'

45 **niveos**] In bran new togas probably given for the occasion.

46 **Defossa**] To make sure that they've got it: also to make sure that they will not lose it, cf. Fallacem circum, Hor. *Sat.* I. vi. 113.

47 **Tum**] Even between B.C. 460—357.

50] An Abderite would have hung himself. The cord giving way, he fell, and broke his head. He first went to the surgeon, and had his wound plastered, and then again hung himself.

[*Specimen Page, No.* 8.]

THE ELECTRA OF

ΗΛ. [*interrupting*] τί τῶν ἀπόντων ἢ τί τῶν ὄντων πέρι;
ΠΡ. [*solemnly*] λαβεῖν φίλον θησαυρόν, ὃν φαίνει θεός·
ΗΛ. ἰδού, καλῶ θεούς.
[*clasping her hands*] ἦ τί δὴ λέγεις, γέρον;
ΠΡ. βλέψον νυν ἐς τόνδ᾽, ὦ τέκνον, τὸν φίλτατον.
[*turning her round to* ORESTES.]
ΗΛ. [*sadly*] πάλαι δέδοικα, μὴ σύ γ᾽ οὐκέτ᾽ εὖ φρονῇς·
ΠΡ. οὐκ εὖ φρονῶ ᾽γὼ σὸν κασίγνητον βλέπων;
ΗΛ. [*starting suddenly*]
πῶς εἶπας, ὦ γεραί᾽, ἀνέλπιστον λόγον;
ΠΡ. [*emphatically*] ὁρᾶν Ὀρέστην τόνδε τὸν Ἀγαμέμνονος.
ΗΛ. ποῖον χαρακτῆρ᾽ εἰσιδών, ᾧ πείσομαι; [*incredulous*]
ΠΡ. [*pointing at a scar in* ORESTES᾽ *forehead*]
οὐλὴν παρ᾽ ὀφρύν, ἥν ποτ᾽ ἐν πατρός δόμοις
νεβρὸν διώκων σοῦ μέθ᾽ ᾑμάχθη πεσών.
ΗΛ. πῶς φῄς; ὁρῶ μὲν πτώματος τεκμήριον·
[*astounded, but still hesitating.*]
ΠΡ. ἔπειτα μέλλεις προσπίτνειν τοῖς φιλτάτοις;
ΗΛ. [*resolved*] ἀλλ᾽ οὐκέτ᾽, ὦ γεραιέ· συμβόλοισι γὰρ
τοῖς σοῖς πέπεισμαι θυμόν. [*she rushes in a transport of joy into her brother's arms.*] ὦ χρόνῳ φανείς,
ἔχω σ᾽ ἀέλπτως. ΟΡ. κἀξ ἐμοῦ γ᾽ ἔχει χρόνῳ.
ΗΛ. οὐδέποτε δόξασ᾽. ΟΡ. οὐδ᾽ ἐγὼ γὰρ ἤλπισα.
ΠΡ. ἐκεῖνος εἶ σύ;
ΟΡ. σύμμαχός γέ σοι μόνος,
ἦν ἐκσπάσωμαί γ᾽ ὃν μετέρχομαι βόλον.
πέποιθα δ᾽. ἢ χρὴ μηκέθ᾽ ἡγεῖσθαι θεούς,
εἰ τἄδικ᾽ ἔσται τῆς δίκης ὑπέρτερα. [*with confidence.*]

[*Scenes from* Greek Plays. See page 18.]

EXERCISE XXII.

HERCULES.

I was born a boy, stronger than brother Ĭphicles,
a new-born babe worthy of Zeus as father;
and I showed strength, released from swaddling clothes;
and I proved myself to all nobly bred.
5 Hērā sent on us two two snakes for murder;
and just before dawn flashed down a dreadful light on the bed.
Iphicles seeing monsters weeps in vain,
and silently crouches hidden in bed-clothes;
but I shouted aloud having conquered serpents:
10 and this is first of contests. And the neighbours
asked, How is Ampĭtryon father of the boy?
for he prevails over hydra and savage lion;
running, not hunting, he catches a stag,

---

1. *I was born*, Ex. v. 8.

2. *New-born*, *νεογνὸς.*

3. *To release*, *ἀπαλλάσσειν.*

4. *Proved myself*, aor. pass. of *φαίνω*. *Bred*, perf. part. Anapæst in first foot, or tribrach in second.

5. *Two*, sign of the dual. *For*, *πρὸς.*

6. *Just before*, *ὑπὸ* with the accusative. *To flash down on*, *κατασκήπτω.*

7. *Monster*, *δᾰ́κος.* *In vain*, Ex. xvi. 6. Insert *μὲν* for the sake of contrast with the ninth line, as in Ex. xix. 1.

9. *To shout aloud*, *ἀναλαλάζειν.*

11. *To ask a question*, *ἐρωτᾶν*: aorist, *ἠρόμην*. The three last syllables of *Ἀμφιτρύων* make an anapest.

12. *To prevail over*, *κρατεῖν*, with the genitive.

13. *Running*, *δρομαῖος.* *To hunt* = to be a hunter. static verb from *κυνηγέτης.* Tribrach in third foot.

[*Iophon.* *See page* 17.]

[*Specimen Page, No.* 10.]

their kind, and of every creeping thing of the earth after his kind." Sufficient food was also to be provided: "take thou unto thee of all food that is eaten, and thou shalt gather it to thee, and it shall be for food for thee and for them" [GEN. vi. 19-21].

To make all these preparations required a strong belief in God on the part of Noah. The world around him utterly disbelieved the message which he conveyed to it during many years of preparation as the "preacher of righteousness" [2 PET. ii. 5], while God's longsuffering waited [1 PET. iii. 20]. Our Lord says that "they were eating and drinking, marrying and giving in marriage, until the day that Noah entered into the ark, and knew not until the flood came and took them all away" [MATT. xxiv. 38; LUKE xvii. 26]. But though all the world disregarded, Noah was entitled to be enrolled among the number of St. Paul's "elders who obtained a good report," for his faith made him believe in the things of which God gave him warning "though not seen as yet" [HEB. xi. 7], and it is recorded of him, "Thus did Noah; according to all that God commanded him so did he" [GEN. vi. 22].

The Ark which Noah built in obedience to the Divine command was not a navigable ship, but a great wooden "coffer," or water-tight chest, made so as to float about steadily upon the water.[1]

It was built of cypress or "gopher" wood, and covered with pitch within and without to secure it against leakage from the flood below or the rain above. The size of the ark is distinctly given as being 300 cubits in length by 50 cubits in width, and 30 cubits in height. The cubit is reckoned at about 21 inches, and we are thus able to compare the size of the ark with that of our large iron and wooden ships of modern days.[2]

	Length.	Breadth.	Depth.
The Ark	525 feet	87 feet 6 inches	52 feet 6 inches
Duke of Wellington	240 feet	60 feet	72 feet 4 inches
Great Eastern	680 feet	83 feet	58 feet

---

[1] Its object being the same as that of the "ark" in which the infant Moses was placed when cast into the Nile in obedience to the edict of Pharaoh.

[2] The proportions of the ark are exactly those of the human body, viz., 10·+1·6+1·; and the capacity of these proportions for stowage has been proved by experiments in Holland and Denmark to be a third greater than that of vessels as built for ordinary sailing purposes. That of the Ark was thus about the same as that of the Great Eastern.

[*Companion to Old Testament. See page* 24.]

[*Specimen Page, No.* 11.]

Twenty-ninth Lesson.

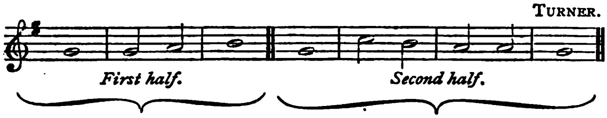
CHANTING.

CHANTING is the arrangement of prose in a rhythmical form. The psalms, canticles, &c. are sung or chanted to melodies called CHANTS, which are either SINGLE or DOUBLE.

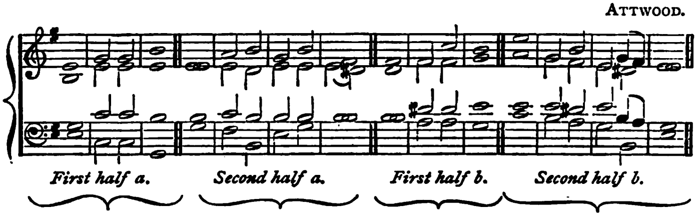
The melody of a single chant is, for convenience, written in phrases of seven bars of two minims each or their value.

The first half of a chant has three, the second four bars.

The first half is called the *mediation*, the second the *cadence.*



A double chant is simply a single chant form repeated.



A single chant is arranged to fit one verse of the psalms, a double chant two; for the long psalms quadruple chants, of which the phrase or melody is designed to include four verses, have been written.

A changeable chant is one whose key-chord may be either

[*The Chorister's Guide. See page* 28.]

(especially in winter), and only a limited number of troops can march along one road. Thus all roads leading out of a fortress are to some extent like causeways across a marsh, for practical purposes. The difficulty is diminished by acting at night, and by making feints.

24. Fort St. Georges was on the east, La Favorita on the north side, both on the outside of the lakes. A tête-de-chaussée is a fort which commands and "caps" a road, as a tête-de-pont does a bridge.

25. "Considered himself able to obtain."

26. Detached, that is, from the army now under the Archduke Charles. Till this new force, under a new general, should arrive, Melas was left in command of what remained of Beaulieu's army, now in retreat up the valley of the Adige. Beaulieu himself was recalled.

27. The district called the Vorarlberg lies between the Lake of Constance and the Tyrol. The Tyrolese attachment to the House of Austria is famous. In 1809, Napoleon wanted to take the Tyrol from Austria, and give it to Bavaria, setting up the latter as a rival power to Austria. The Tyrolese resisted. [Story of Hofer.]

28. [Why did not Bonaparte cross the Adige, or else ascend it, and make for the Danube?]

29. "Dependent on" (comp. the English "irrelevant") . . . "invested with," i.e. holding. These little domains were only nominally dependent on the empire; in reality they were part of the territory of Genoa, and contributed to its militia. "The empire" had only eight years more to live. When Francis II. saw that he had lost all real power as emperor, he threw it up altogether, and took the title of Emperor of Austria instead.

30. [St. Januarius.]

31. There were also six thousand English in Corsica, who might have reinforced an army attacking Bonaparte from the south. [Have English troops ever been in North Italy? Only once, I believe.]

32. In its lower course, the Po is higher than the surrounding country, thanks to the deposits brought down from the Alps, which raise its bed incessantly. It is walled in by high embankments, kept in order by a staff of engineers, as in Holland. But, in spite of their efforts, the river sometimes breaks through.

33. "Referred the question of peace to."

34. Napoleon had strange good fortune in one respect: his enemies never attacked him at the same moment. In this campaign he could hardly have resisted a flank attack from a Papal and Neapolitan army combined with that of the Austrians. So, when he beat Austria at Austerlitz, Prussia on his left flank was holding back; when he beat Prussia at Jena, Austria on his right flank was passive; when he invaded Russia, neither Prussia nor Austria stirred; when at last they did combine in one attack, they were more than a match for him, and he was ruined in the great battle of 1813.

[*Campaigns of Napoleon.* See page 27.]

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